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THEORY OF LAMINATED ELASTIC PLATES
I. ISOTROPIC LAMINAE

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CONTENTS

In this paper (part I) we establish a theory for stretching and bending of laminated elastic plates in which the laminae are different isotropic linearly elastic materials. The theory gives exact solutions of the three-dimensional elasticity equations that satisfy all the interface traction and displacement continuity conditions, with no traction on the lateral surfaces; the only restriction is that edge boundary conditions can be satisfied only in an average manner, rather than point by point. The method, which is based on a generalization of Michell's exact plane stress theory, yields exact solutions for each lamina. These solutions are generated in a very straightforward manner by solutions of the approximate two-dimensional classical equations of

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laminate theory and contain sufficient arbitrary constants to enable all the continuity and lateral surface boundary conditions to be satisfied. The values of the constants depend only on the lamina thicknesses and the elastic constants. Thus, for a given laminate and for any boundary-value problem, it is necessary only to solve the appropriate two-dimensional plane problem, and the corresponding exact three-dimensional laminate solution follows by straightforward substitutions. The two-dimensional solution may be derived by any of the available methods, including numerical methods. An important feature of the theory is that it determines the interfacial shearing tractions, as well as the in-plane stress components. The procedure is illustrated by applying the theory to three problems involving stretching and bending of laminated plates containing circular holes.

1. Introduction

The use of laminated plates as structural members is not new; the use of plywood is an obvious example of a familiar application. However, the stress and deformation analysis of laminated plates has acquired greatly increased interest in recent years because many new and advanced materials, such as fibre-reinforced composite materials, are frequently and conveniently used in laminated form. This is particularly true of applications in the aerospace industries, where it is important to take maximum advantage of the high specific strength and stiffness, with consequent mass-saving ability, which the new materials make available.

The most commonly used approach to stress analysis of elastic laminated plates is known as classical laminate theory, and is described by, for example, Christensen (1979). In this theory the heterogeneous laminated plate is replaced by an equivalent plate of the same overall geometry and with elastic constants which are appropriate weighted averages of the elastic constants of the laminae. This theory is usually found to give satisfactory results for the deflection of the mid-surface of the plate, and for the average in-plane displacement and stress components. However, there is a need for quantitative assessment of the errors involved, and it is certainly the case that, because it is based on an averaging procedure, classical laminate theory can yield only limited information about the through-thickness distribution of stress and deformation, and no information at all about the interlaminar shear forces. In a laminate, the in-plane stress and strain components are often subject to very large variations through the thickness, and so the use of averages can conceal many significant effects in the individual laminae. Furthermore, the interlaminar shear tractions have an important influence on the onset of delamination and consequent failure of the laminate, so knowledge of these tractions is also important. We therefore require a more refined theory of laminates than is provided by classical laminate theory.

Several such theories have been put forward. In one such class of theories (termed higher-order theories), such as those proposed by Whitney & Pagano (1970), Whitney (1972), Nelson & Lorch (1974) and Lo *et al.* (1977), displacement components are approximated through the plate thickness by polynomials in Z, the coordinate normal to the plate. This approach seems to us to be inherently unsatisfactory in that it fails to take into account the essential fact that the displacement gradients with respect to Z are usually discontinuous at the interlaminar interfaces. Consequently, except in elementary cases, these theories fail to satisfy shear traction continuity conditions at the interfaces. Comparison with the exact elasticity solutions for some simple problems (Bonser 1984) confirms that the theory of Lo *et al.* (1977) often gives poor agreement with the exact solutions.

A more satisfactory theory has been proposed by Srinivas (1973). In this theory the in-plane displacement components are assumed to be piecewise linear in Z, with discontinuous derivatives with respect to Z at the interlaminar interfaces. In test problems Bonser (1984) shows that this theory also gives results which are often substantially in error. In particular, this theory does not, in general, admit continuity of the shear traction at interlaminar interfaces, nor does it satisfy zero shear traction conditions on the lateral surfaces.

A further theory is due to Pagano (1978). In this the in-plane stress components are taken to be piecewise linear in Z. This theory admits continuity of displacement and traction at the interfaces, but equilibrium equations are satisfied only in integrated form for each lamina. In test problems this theory also gives results which depart considerably from those given by exact three-dimensional solutions. We conclude that none of the existing laminate theories known to us can be regarded as providing a satisfactory means of analysing the variation of both stress and deformation throughout the thickness of a laminate, nor of determining the interfacial shear tractions between laminae.

In this paper we present an elasticity theory for isotropic laminated plates which, in a very simple manner, yields exact solutions of the three-dimensional elasticity equations, satisfies all traction and displacement continuity conditions at the interlaminar interfaces, and satisfies zero traction conditions on the lateral surfaces, all for a laminate composed of any number of laminae. The only respect in which any form of approximation is involved is one which is common to all plate theories in that edge boundary conditions are specified in terms of the average through-thickness displacement or of the stress and moment resultants, rather than by pointwise specification of edge displacement or traction. Thus, according to Saint-Venant's principle, the solutions are valid everywhere except in edge boundary layers whose width is of the order of the plate thickness.

In this paper (part I) the analysis is restricted to laminates whose laminae are of different isotropic elastic materials. Although such configurations are of interest in connection with, for example, sandwich plates, they clearly fail to address the main practical problem in this area, which is the analysis of laminates whose constituent layers are anisotropic. In particular, interest centres on the case in which the laminae are differently oriented sheets of uniaxially fibre-reinforced materials, which may be modelled as transversely isotropic or orthotropic laminae. However, the analysis presented here for isotropic laminae gives strong indications of the manner in which we should proceed with the analysis of anisotropic laminae, and such an analysis forms the subject of part II (in preparation). The solutions, being exact, are also of obvious importance as tests of numerical procedures for the stress analysis of laminates.

The theory that we present is in the spirit of the exact plane stress theory for stretching and bending of moderately thick homogeneous plates which was formulated by Michell (1900) and described in some detail by Love (1927, articles 300–304). Further developments of this theory have been described by Lur'e (1964) and by Reiss & Locke (1961). Our theory may be regarded as an extension, to heterogeneous laminated media, of the exact plane stress theory.

In §2 we consider the stress and deformation in a single layer of the laminate. After a brief outline of some necessary results in the two-dimensional classical thin-plate theory, we derive a generalization of the exact solutions given by Love (1927) for an homogeneous layer. This generalization comprises an exact solution of the three-dimensional elasticity equations; it contains a number of disposable arbitrary constants and does not require the shear tractions on the lateral surfaces of the layer to vanish. The solution decomposes into two independent solutions (the stretching solution and the bending solution) which we treat separately. Our method of derivation differs from that of Love (1927) but has features in common with a method used by Reiss & Locke (1961). An essential feature of the solution is that, to establish an exact three-dimensional solution, it is necessary only to solve the two-dimensional thin-plate equations; any solution of the two-dimensional equations immediately generates, by simple substitution, a solution of the three-dimensional equations which contains a number of arbitrary constants. In the more restricted context of Michell's solutions, this result is implicit in the account given by Love (1927), but does not seem to have been explicitly stated or exploited.

The classical laminate theory mentioned above is outlined in §3. In particular we define here the equivalent plate for a laminate. This is a homogeneous plate, of the same overall dimensions as the laminate, and with elastic constants which are appropriate averages of the elastic constants of the laminae. Classical laminate theory in effect applies thin-plate theory to the equivalent plate.

In §4 we establish the exact laminate theory. We use the two-dimensional classical laminate theory, based on the equivalent plate, to generate exact three-dimensional solutions in each layer of the laminate. These solutions contain several arbitrary constants for each layer; it is found that these suffice to satisfy all the interface traction and displacement continuity conditions, and vanishing traction conditions on the lateral surfaces. Thus ultimately, and in a very direct and straightforward manner, we find that any solution of the two-dimensional classical laminate theory equations generates a corresponding full, exact three-dimensional elasticity solution for the entire laminate. There is no restriction on the number of laminae. The values of the arbitrary constants do not depend on the problem under consideration, but only on the geometrical and elastic properties of the laminate, so that for a given laminate they need be evaluated once only. The underlying solution of the classical laminate theory equations may be obtained by any of the various methods available for solving plane elastic problems, including numerical methods.

Some illustrative applications of the theory are described in §5. These concern infinite plates containing circular traction-free or clamped holes, in uniaxial tension and in bending. Section 6 contains discussion of various aspects of the theory.

2. Deformation and stress in a single layer

(a) *Notation and general theory*

We consider a layer of uniform thickness of isotropic, linearly elastic material. Initially we use rectangular cartesian coordinates X, Y, Z such that $Z = 0$ coincides with the mid-plane of the layer. Referred to these coordinates, the components of displacement are denoted by U, V, W, and the components of the symmetric stress tensor $\boldsymbol{\sigma}$ by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{2.1}$$

The Lamé elastic constants are denoted by λ and μ, so that the stress–strain relations can be expressed as

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda\\ \lambda & \lambda+2\mu & \lambda\\ \lambda & \lambda & \lambda+2\mu\end{bmatrix}\begin{bmatrix}U_{,X}\\ V_{,Y}\\ W_{,Z}\end{bmatrix}, \tag{2.2}$$

$$\begin{bmatrix}\sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{bmatrix} = \mu\begin{bmatrix}V_{,Z}+W_{,Y}\\ W_{,X}+U_{,Z}\\ U_{,Y}+V_{,X}\end{bmatrix}, \tag{2.3}$$

where commas denote partial differentiation with respect to the suffix variables.

We require solutions for which the normal stress σ_{zz} is zero at the lateral surfaces of the layer. Because we shall also require the possibility of non-zero shear tractions at these surfaces, then plane-stress or generalized plane-stress conditions (e.g. Love 1927) cannot be assumed. Instead, we seek solutions for which we assume only that

$$\sigma_{zz} = 0 \tag{2.4}$$

holds everywhere in the layer. In this case

$$\lambda(U_{,X}+V_{,Y})+(\lambda+2\mu)\,W_{,Z} = 0, \tag{2.5}$$

and (2.2) reduces to

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\end{bmatrix} = \begin{bmatrix}\lambda'+2\mu & \lambda'\\ \lambda' & \lambda'+2\mu\end{bmatrix}\begin{bmatrix}U_{,X}\\ V_{,Y}\end{bmatrix}, \tag{2.6}$$

where (following the notation of Love (1927, p. 208))

$$\lambda' = 2\lambda\mu/(\lambda+2\mu).$$

Thus λ' and μ are effective elastic moduli for the case $\sigma_{zz} = 0$. We also introduce the dimensionless elastic constant

$$\eta = \lambda'/2\mu. \tag{2.7}$$

The constant η is related to Poisson's ratio ν by

$$\eta = \nu/(1-\nu). \tag{2.8}$$

Thus (2.6) may be re-written as

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\end{bmatrix} = 2\mu\begin{bmatrix}\eta+1 & \eta\\ \eta & \eta+1\end{bmatrix}\begin{bmatrix}U_{,X}\\ V_{,Y}\end{bmatrix}, \tag{2.9}$$

and (2.5) as

$$\eta(U_{,X}+V_{,Y})+W_{,Z} = 0. \tag{2.10}$$

With $\sigma_{zz} = 0$, and assuming body forces to be zero, the equations of equilibrium reduce to

$$\left.\begin{aligned}\sigma_{xx,X}+\sigma_{xy,Y}+\sigma_{xz,Z} &= 0,\\ \sigma_{xy,X}+\sigma_{yy,Y}+\sigma_{yz,Z} &= 0,\\ \sigma_{xz,X}+\sigma_{yz,Y} &= 0.\end{aligned}\right\} \tag{2.11}$$

(b) *Classical thin-plate theory*

We shall seek certain exact solutions of the three-dimensional equations (2.9)–(2.11). However, first it is necessary to outline some results of classical thin-plate theory, which is an approximate theory.

We consider a plate of thickness $2H$ whose mid-plane lies in the plane $Z = 0$. Whereas variations of the field variables in the X and Y directions may be considered in terms of a typical in-plane linear dimension a, which is usually defined by a given boundary-value problem, variations with respect to Z are naturally related to H, where normally $H \ll a$. Accordingly we introduce non-dimensional variables x, y, z, u, v and w, defined through

$$\left.\begin{aligned} x = X/a, \quad y = Y/a, \quad z = Z/H, \\ u = U/a, \quad v = V/a, \quad w = W/a. \end{aligned}\right\} \tag{2.12}$$

We also introduce the dimensionless constant

$$\epsilon = H/a. \tag{2.13}$$

In thin plate theory the scaled displacement components $u(x,y,z)$ and $v(x,y,z)$ are approximated as

$$\left.\begin{aligned} u(x,y,z) \approx \bar{u}(x,y) - \epsilon z \bar{w}(x,y)_{,x}, \\ v(x,y,z) \approx \bar{v}(x,y) - \epsilon z \bar{w}(x,y)_{,y}, \end{aligned}\right\} \tag{2.14}$$

where $\bar{u}(x,y)$ and $\bar{v}(x,y)$ are the through-thickness averages of u and v

$$\{\bar{u}(x,y), \bar{v}(x,y)\} = \frac{1}{2}\int_{-1}^{1} \{u(x,y,z), v(x,y,z)\}\,\mathrm{d}z \tag{2.15}$$

and $\bar{w}(x,y) = w(x,y,0)$ is the transverse displacement of the middle surface. Geometrically (2.14) state that plane sections that are initially normal to the middle surface remain plane and normal to the middle surface. It is also assumed that the lateral surfaces of the plate are free from traction.

Stress resultants are defined by

$$(N_{xx}, N_{yy}, N_{xy}, N_{xz}, N_{yz}) = H\int_{-1}^{1} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})\,\mathrm{d}z. \tag{2.16}$$

By substituting (2.14) into the stress–strain relations and integrating from $z = -1$ to $z = 1$, we obtain

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = 2\hat{\mu}H \begin{bmatrix} 2(\hat{\eta}+1) & 2\hat{\eta} & 0 \\ 2\hat{\eta} & 2(\hat{\eta}+1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_{,x} \\ \bar{v}_{,y} \\ \bar{u}_{,y} + \bar{v}_{,x} \end{bmatrix}, \tag{2.17}$$

where
$$(\hat{\lambda}', \hat{\mu}) = \frac{1}{2}\int_{-1}^{1} \{\lambda'(z), \mu(z)\}\,\mathrm{d}z, \quad \hat{\eta} = \hat{\lambda}'/2\hat{\mu}. \tag{2.18}$$

Thus $\hat{\lambda}'$ and $\hat{\mu}$ represent average values of λ' and μ through the plate thickness. In the case of a homogeneous plate, $\hat{\lambda}' = \lambda'$, $\hat{\mu} = \mu$ and therefore $\hat{\eta} = \eta$.

By integrating the first two equilibrium equations from $z = -1$ to $z = 1$, and imposing the condition that the lateral surfaces are traction-free, we obtain

$$N_{xx,x} + N_{xy,y} = 0, \quad N_{xy,x} + N_{yy,y} = 0. \tag{2.19}$$

Then by substituting (2.17) into (2.19) it follows that

$$2(\hat{\eta}+1)\Delta_{,x}-\Omega_{,y}=0,\quad 2(\hat{\eta}+1)\Delta_{,y}+\Omega_{,x}=0, \tag{2.20}$$

where
$$\Delta=\bar{u}_{,x}+\bar{v}_{,y},\quad \Omega=\bar{v}_{,x}-\bar{u}_{,y}. \tag{2.21}$$

By eliminating Δ and Ω in turn from (2.20), we obtain

$$\nabla^2\Delta=0,\quad \nabla^2\Omega=0, \tag{2.22}$$

where ∇^2 represents the two-dimensional laplacian operator

$$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2. \tag{2.23}$$

From (2.20), (2.21) and (2.22) it follows that

$$\nabla^4\bar{u}=0,\quad \nabla^4\bar{v}=0. \tag{2.24}$$

The bending moments are defined as

$$(M_{xx},M_{yy},M_{xy})=\int_{-H}^{H}(\sigma_{xx},\sigma_{yy},\sigma_{xy})\,Z\,\mathrm{d}Z=H^2\int_{-1}^{1}(\sigma_{xx},\sigma_{yy},\sigma_{xy})\,z\,\mathrm{d}z. \tag{2.25}$$

Hence from (2.3), (2.9) and (2.14) we obtain

$$\begin{bmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{bmatrix}=-\tfrac{2}{3}\epsilon\tilde{\mu}H^2\begin{bmatrix}2(\tilde{\eta}+1)&2\tilde{\eta}&0\\2\tilde{\eta}&2(\tilde{\eta}+1)&0\\0&0&1\end{bmatrix}\begin{bmatrix}\bar{w}_{,xx}\\\bar{w}_{,yy}\\2\bar{w}_{,xy}\end{bmatrix}, \tag{2.26}$$

where
$$\{\tilde{\lambda}',\tilde{\mu}\}=\frac{3}{2}\int_{-1}^{1}\{\lambda'(z),\mu(z)\}z^2\,\mathrm{d}z,\quad \tilde{\eta}=\tilde{\lambda}'/2\tilde{\mu}. \tag{2.27}$$

For a homogeneous plate, $\tilde{\lambda}'=\lambda'$, $\tilde{\mu}=\mu$ and $\tilde{\eta}=\eta$. By multiplying $(2.11)_{1,2}$ by z, integrating from $z=-1$ to $z=1$, and applying traction-free boundary conditions at $z=\pm1$, we obtain

$$\epsilon(M_{xx,x}+M_{xy,y})-HN_{xz}=0,\quad \epsilon(M_{xy,x}+M_{yy,y})-HN_{yz}=0. \tag{2.28}$$

Also, by integrating $(2.11)_3$ from $z=-1$ to $z=1$, we have

$$N_{xz,x}+N_{yz,y}=0. \tag{2.29}$$

Hence, from (2.28) and (2.29),

$$M_{xx,xx}+2M_{xy,xy}+M_{yy,yy}=0, \tag{2.30}$$

and it follows from (2.26) that

$$\nabla^4\bar{w}=0. \tag{2.31}$$

Equations (2.20) and (2.31) are the basic equations of classical thin-plate theory. When $\bar{u}$, $\bar{v}$ and $\bar{w}$ are determined, the stress resultants N_{xx}, N_{yy} and N_{xy} are determined by (2.17) and the bending moments M_{xx}, M_{yy} and M_{xy} by (2.26). Typical boundary conditions are that at the edge of a plate with outward unit normal $\boldsymbol{n}=(n_x,n_y,0)$ we may specify $\bar{w}$ and one from each of the following pairs:

$$\left.\begin{array}{llll}(a)\ n_x\bar{u}+n_y\bar{v} & \text{or} & (b)\ n_xN_{xx}+n_yN_{xy},\\ (c)\ -n_y\bar{u}+n_x\bar{v} & \text{or} & (d)\ n_xN_{xy}+n_yN_{yy},\\ (e)\ n_x\bar{w}_{,x}+n_y\bar{w}_{,y} & \text{or} & (f)\ n_xM_{xx}+n_yM_{xy}.\end{array}\right\} \tag{2.32}$$

Here (*a*) and (*c*) correspond to the specification of the mean in-plane edge displacement components; (*b*) and (*d*) to the specification of the in-plane components of edge force; (*e*) to the specification of the slope of the mid-surface; and (*f*) to the specification of the bending moments applied to the edge of the plate. Other types of boundary conditions may also arise; see, for example, Timoshenko & Woinowsky-Krieger (1959).

(*c*) *Analysis of stress and deformation in a homogeneous layer*

In this section we consider the stress and deformation in a single layer of homogeneous material, so that λ' and μ are constants within the layer. We make a distinction between a homogeneous layer and a plate, which may be an assemblage of a number of layers of different materials. Accordingly, we denote the uniform thickness of the layer by $2h$ and, within the layer, we introduce non-dimensional variables x, y, $\bar{z}$, u, v and w by

$$\left.\begin{aligned} x &= X/a, \quad y = Y/a, \quad \bar{z} = Z/h, \\ u &= U/a, \quad v = V/a, \quad w = W/a, \end{aligned}\right\} \tag{2.33}$$

where, as before, a is a typical in-plane linear dimension. The plane $Z = 0$ (or $\bar{z} = 0$) is chosen to coincide with the mid-plane of the layer, and the lateral surfaces of the layer are $\bar{z} = \pm 1$. We also define the dimensionless parameter $\bar{\epsilon}$ as

$$\bar{\epsilon} = h/a. \tag{2.34}$$

Although in practice $\bar{\epsilon}$ will usually be small compared with unity, the analysis that follows is not an asymptotic analysis, and is valid for any value of $\bar{\epsilon}$.

In terms of the non-dimensional variables, the governing equations (2.3) and (2.9)–(2.10) take the forms

$$\left.\begin{aligned} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= \mu \begin{bmatrix} 2(\eta+1) & 2\eta & 0 \\ 2\eta & 2(\eta+1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{bmatrix}, \\ \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} &= \mu \begin{bmatrix} \bar{\epsilon}^{-1} u_{,\bar{z}} + w_{,x} \\ \bar{\epsilon}^{-1} v_{,\bar{z}} + w_{,y} \end{bmatrix}, \end{aligned}\right\} \tag{2.35}$$

$$\eta(u_{,x}+v_{,y}) + \bar{\epsilon}^{-1} w_{,\bar{z}} = 0, \tag{2.36}$$

and the equations of equilibrium (2.11) become

$$\left.\begin{aligned} \sigma_{xx,x} + \sigma_{xy,y} + \bar{\epsilon}^{-1}\sigma_{xz,\bar{z}} &= 0, \\ \sigma_{xy,x} + \sigma_{yy,y} + \bar{\epsilon}^{-1}\sigma_{yz,\bar{z}} &= 0, \\ \sigma_{xz,x} + \sigma_{yz,y} &= 0. \end{aligned}\right\} \tag{2.37}$$

By substituting (2.35) into (2.37) we obtain

$$\left.\begin{aligned} \bar{\epsilon}^2\{2(\eta+1)\, u_{,xx} + (2\eta+1)\, v_{,xy} + u_{,yy}\} + (u_{,\bar{z}\bar{z}} + \bar{\epsilon} w_{,x\bar{z}}) &= 0, \\ \bar{\epsilon}^2\{v_{,xx} + (2\eta+1)\, u_{,xy} + 2(\eta+1)\, v_{,yy}\} + (v_{,\bar{z}\bar{z}} + \bar{\epsilon} w_{,y\bar{z}}) &= 0, \\ \bar{\epsilon}(w_{,xx} + w_{,yy}) + (u_{,x\bar{z}} + v_{,y\bar{z}}) &= 0. \end{aligned}\right\} \tag{2.38}$$

In addition, we express (2.36) as

$$\eta\bar{\epsilon}(u_{,x}+v_{,y})+w_{,\bar{z}} = 0. \tag{2.39}$$

Our objective is to determine exact solutions of (2.38) and (2.39), but for convenience in formulating the theory, and with a view to further developments in part II, we suppose initially that in the layer the displacement can be expressed formally by a power series in $\bar{\epsilon}$, as

$$\begin{Bmatrix} u(x,y,\bar{z},\bar{\epsilon}) \\ v(x,y,\bar{z},\bar{\epsilon}) \\ w(x,y,\bar{z},\bar{\epsilon}) \end{Bmatrix} = \sum_{n=0}^{\infty} \bar{\epsilon}^n \begin{Bmatrix} u_n(x,y,\bar{z}) \\ v_n(x,y,\bar{z}) \\ w_n(x,y,\bar{z}) \end{Bmatrix}. \tag{2.40}$$

We also adopt the notations

$$\Delta_n = u_{n,x}+v_{n,y}, \quad \Omega_n = v_{n,x}-u_{n,y}. \tag{2.41}$$

We now substitute (2.40) into (2.38) and (2.39) and equate coefficients of powers of $\bar{\epsilon}$. The terms independent of $\bar{\epsilon}$ give

$$w_{0,\bar{z}} = 0, \quad u_{0,\bar{z}\bar{z}} = 0, \quad v_{0,\bar{z}\bar{z}} = 0, \quad u_{0,x\bar{z}}+v_{0,y\bar{z}} = 0. \tag{2.42}$$

These are satisfied if we choose u_0, v_0 and w_0 to be independent of $\bar{z}$. In particular, they are satisfied if u_0, v_0 and w_0 are identified with a solution of the equations of classical thin-plate theory. Accordingly we adopt classical plate theory as a first approximation of the three-dimensional equations (without specifying yet the elastic constants λ' and μ which are to be used), so that u_0, v_0 and w_0 satisfy

$$\nabla^2\Delta_0 = 0, \quad \nabla^2\Omega_0 = 0, \quad \nabla^4 w_0 = 0. \tag{2.43}$$

The terms of order $\bar{\epsilon}$ in the expansion give

$$w_{1,\bar{z}}+\eta\Delta_0 = 0, \quad u_{1,\bar{z}\bar{z}}+w_{0,x\bar{z}} = 0,$$

$$v_{1,\bar{z}\bar{z}}+w_{0,y\bar{z}} = 0, \quad u_{1,x\bar{z}}+v_{1,y\bar{z}}+\nabla^2 w_0 = 0.$$

A solution of these, with (2.42), is

$$u_1 = -(\bar{z}+B_1)\,w_{0,x}, \quad v_1 = -(\bar{z}+B_1)\,w_{0,y}, \quad w_1 = -(\bar{z}+S_1)\,\eta\Delta_0, \tag{2.44}$$

where B_1 and S_1 are constants. This is not the most general solution of the equations, but has sufficient generality for our purposes.

For $n \geqslant 2$, the terms of order $\bar{\epsilon}^n$ give

$$\left.\begin{aligned} \eta\Delta_{n-1}+w_{n,\bar{z}} &= 0, \\ 2(\eta+1)\,\Delta_{n-2,x}-\Omega_{n-2,y}+w_{n-1,x\bar{z}}+u_{n,\bar{z}\bar{z}} &= 0, \\ 2(\eta+1)\,\Delta_{n-2,y}+\Omega_{n-2,x}+w_{n-1,y\bar{z}}+v_{n,\bar{z}\bar{z}} &= 0, \\ \nabla^2 w_{n-1}+\Delta_{n,\bar{z}} &= 0. \end{aligned}\right\} \tag{2.45}$$

For $n = 2$, and using (2.43) and (2.44), these have a solution

$$\left.\begin{aligned} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} &= -(\tfrac{1}{2}\bar{z}^2+S_2\bar{z}+S_3)\left\{(\eta+2)\begin{bmatrix} \Delta_{0,x} \\ \Delta_{0,y} \end{bmatrix}+\begin{bmatrix} -\Omega_{0,y} \\ \Omega_{0,x} \end{bmatrix}\right\}, \\ w_2 &= (\tfrac{1}{2}\bar{z}^2+B_1\bar{z}+B_2)\,\eta\nabla^2 w_0, \end{aligned}\right\} \tag{2.46}$$

where S_2, S_3 and B_2 are a further set of arbitrary constants. For $n = 3$, by (2.43) and (2.46), (2.45) yields a solution

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = (\tfrac{1}{6}\bar{z}^3 + \tfrac{1}{2}B_1\bar{z}^2 + B_3\bar{z} + B_4)(\eta+2)\begin{bmatrix} \nabla^2 w_{0,x} \\ \nabla^2 w_{0,y} \end{bmatrix}, \quad w_3 = 0, \tag{2.47}$$

where B_3 and B_4 are further constants.

It can then be verified that, for $n \geqslant 4$, with u_0, v_0 and w_0 satisfying (2.43), (2.45) is satisfied by

$$u_n = 0, \quad v_n = 0, \quad w_n = 0, \quad n \geqslant 4. \tag{2.48}$$

Consequently the series (2.40) terminates at $n = 3$ for u and v and at $n = 2$ for w. By collecting the terms of the series we have, from (2.44), (2.46) and (2.47), the complete solution

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} - \bar{\epsilon}(\bar{z}+B_1)\begin{bmatrix} w_{0,x} \\ w_{0,y} \end{bmatrix} \\ - \bar{\epsilon}^2(\tfrac{1}{2}\bar{z}^2 + S_2\bar{z} + S_3)\left\{(\eta+2)\begin{bmatrix} \Delta_{0,x} \\ \Delta_{0,y} \end{bmatrix} + \begin{bmatrix} -\Omega_{0,y} \\ \Omega_{0,x} \end{bmatrix}\right\} \\ + \bar{\epsilon}^3(\tfrac{1}{6}\bar{z}^3 + \tfrac{1}{2}B_1\bar{z}^2 + B_3\bar{z} + B_4)(\eta+2)\begin{bmatrix} \nabla^2 w_{0,x} \\ \nabla^2 w_{0,y} \end{bmatrix}, \end{aligned} \tag{2.49}$$

$$w = w_0(x,y) - \bar{\epsilon}\eta(\bar{z}+S_1)\Delta_0 + \bar{\epsilon}^2\eta(\tfrac{1}{2}\bar{z}^2 + B_1\bar{z} + B_2)\nabla^2 w_0. \tag{2.50}$$

It may be verified by direct substitution that (2.49) and (2.50) are an exact closed form solution of the full three-dimensional equations (2.38) and (2.39) provided that Δ_0 and Ω_0 are harmonic functions, and w_0 satisfies the biharmonic equation; that is, provided that u_0, v_0 and w_0 represent a solution of equations (2.20) and (2.31) of classical plate theory for any constant values (not necessarily the actual values) of the elastic constants η and μ. Thus by solving the thin-plate equations it is possible immediately to extend the solution to give an exact solution of the three-dimensional elasticity equations. The arbitrary constants S_1, S_2, S_3, B_1, B_2, B_3 and B_4 allow some freedom in the specification of boundary conditions at the lateral surfaces.

If the lateral surfaces $\bar{z} = \pm 1$ of the layer were to be free from shear traction, so that plane stress conditions applied, then both σ_{xz} and σ_{yz} would be zero on these surfaces. However, we intend in §4 to apply the solution to laminated plates, in which the surfaces of a typical layer will not be traction-free, so we do not impose any restrictions on σ_{xz} and σ_{yz}, and in general allow σ_{xz} and σ_{yz} to be non-zero at the lateral surfaces.

The solution derived in this section is a generalization of solutions originally due to Michell (1900) and described in Love (1927, articles 300–304). Michell assumed that, in addition to $\sigma_{zz} = 0$, the stress components σ_{xz} and σ_{yz} are zero throughout the plate; this leads to a special case of the solution given here. Michell's method of derivation is very different from ours, but it may be extended to give the same results. A third method of derivation is to proceed iteratively, by rearranging (2.38) and (2.39) as iteration formulae and adopting solutions of (2.43) as the first iteration. The iterative process terminates in the exact solution given above.

The solution (2.49) and (2.50) can be decomposed into the sum of two independent solutions, which we term the stretching and the bending solutions respectively. In the stretching solution we have $w_0 = 0$; in the bending solution we have $u_0 = 0$, $v_0 = 0$. The two solutions may of course be superposed. In anticipation of this decomposition, the arbitrary constants have been denoted so that S_1, S_2 and S_3 relate to the stretching solution, and B_1, B_2, B_3 and B_4 relate to the bending solution. Henceforth we deal with the two solutions separately.

(*d*) *The stretching solution for a layer*

In this section we consider the stretching solution, for which (2.49) and (2.50) give the displacement

$$\left.\begin{aligned}\begin{bmatrix}u\\v\end{bmatrix} &= \begin{bmatrix}u_0(x,y)\\v_0(x,y)\end{bmatrix} - \bar{\epsilon}^2(\tfrac{1}{2}\bar{z}^2+S_2\bar{z}+S_3)\left\{(\eta+2)\begin{bmatrix}\Delta_{0,x}\\\Delta_{0,y}\end{bmatrix}+\begin{bmatrix}-\Omega_{0,y}\\\Omega_{0,x}\end{bmatrix}\right\},\\ w &= -\bar{\epsilon}\eta(\bar{z}+S_1)\,\Delta_0,\end{aligned}\right\} \tag{2.51}$$

and the corresponding stress is

$$\left.\begin{aligned}\frac{1}{\mu}\begin{bmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{bmatrix} &= 2\eta\begin{bmatrix}\Delta_0\\\Delta_0\\0\end{bmatrix}+\begin{bmatrix}2u_{0,x}\\2v_{0,y}\\u_{0,y}+v_{0,x}\end{bmatrix}-\bar{\epsilon}^2(\tfrac{1}{2}\bar{z}^2+S_2\bar{z}+S_3)\\ &\quad\times\left\{2(\eta+2)\begin{bmatrix}\Delta_{0,xx}\\\Delta_{0,yy}\\\Delta_{0,xy}\end{bmatrix}+\begin{bmatrix}-2\Omega_{0,xy}\\2\Omega_{0,xy}\\\Omega_{0,xx}-\Omega_{0,yy}\end{bmatrix}\right\},\\ \frac{1}{\mu}\begin{bmatrix}\sigma_{xz}\\\sigma_{yz}\end{bmatrix} &= -\bar{\epsilon}\{2(\eta+1)\,\bar{z}+S_1\eta+S_2(\eta+2)\}\begin{bmatrix}\Delta_{0,x}\\\Delta_{0,y}\end{bmatrix}-\bar{\epsilon}(\bar{z}+S_2)\begin{bmatrix}-\Omega_{0,y}\\\Omega_{0,x}\end{bmatrix},\\ \sigma_{zz} &= 0.\end{aligned}\right\} \tag{2.52}$$

We note that if it is required that generalized plane stress conditions hold, so that both σ_{xz} and σ_{yz} are zero at the surfaces $\bar{z} = \pm 1$, then

$$2(\eta+1)\,\Delta_{0,x}-\Omega_{0,y} = 0, \quad 2(\eta+1)\,\Delta_{0,y}+\Omega_{0,x} = 0, \tag{2.53}$$

and hence $$S_1 = S_2.$$

Thus in this case, by comparing with (2.20), we see that u_0 and v_0 are solutions of the classical thin-plate equations for the layer. If furthermore the mid-plane $\bar{z} = 0$ is to remain in the plane of symmetry then $S_1 = 0$ and hence $S_2 = 0$. If finally we require the average displacement to coincide with that of classical thin plate theory, then $S_3 = -\frac{1}{6}$, and the solution is reduced to the exact plane stress solution given in Love (1927, article 301).

Obviously the solution (2.51) and (2.52) is considerably more general than that given by Love, not only because it includes the additional arbitrary constants S_1, S_2 and S_3, but also because in it u_0 and v_0 are required only to satisfy the equations

$$\nabla^2\Delta_0 = 0, \quad \Delta^2\Omega_0 = 0. \tag{2.54}$$

These are necessary consequences of (2.53), but are less restrictive. This additional generality is crucial to the solutions for laminated plates which follow in §4.

We note also that a solution of (2.54) (or of (2.53)), which involve only the two independent variables x and y, immediately generates an exact solution (2.51) and (2.52) of the full three-dimensional plate equations.

The solution (2.51) and (2.52) does not allow the pointwise specification of the displacement, or of the edge traction, at the edges of the layer. However, it is possible to specify average values of u and v, or the normal and tangential components of the resultant edge traction, namely

$$n_x N_{xx} + n_y N_{xy} \quad \text{and} \quad n_x N_{xy} + n_y N_{yy}. \tag{2.55}$$

(*e*) *The bending solution for a layer*

In the bending solution, putting $u_0 = 0$ and $v_0 = 0$ into (2.49) and (2.50) gives the displacement as

$$\left.\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= -\bar{\epsilon}(\bar{z}+B)_1) \begin{bmatrix} w_{0,x} \\ w_{0,y} \end{bmatrix} + \bar{\epsilon}^3 (\tfrac{1}{6}\bar{z}^3 + \tfrac{1}{2}B_1 \bar{z}^2 + B_3 \bar{z} + B_4)(\eta+2) \begin{bmatrix} \nabla^2 w_{0,x} \\ \nabla^2 w_{0,y} \end{bmatrix}, \\ w &= w_0(x,y) + \bar{\epsilon}^2 \eta (\tfrac{1}{2}\bar{z}^2 + B_1 \bar{z} + B_2) \nabla^2 w_0. \end{aligned}\right\} \tag{2.56}$$

The corresponding stress is

$$\left.\begin{aligned} \frac{1}{\mu}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= -2\bar{\epsilon}(\bar{z}+B_1)\left\{\eta \begin{bmatrix} \nabla^2 w_0 \\ \nabla^2 w_0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_{0,xx} \\ w_{0,yy} \\ w_{0,xy} \end{bmatrix}\right\} \\ &\quad + 2\bar{\epsilon}^3(\tfrac{1}{6}\bar{z}^3 + \tfrac{1}{2}B_1\bar{z}^2 + B_3\bar{z} + B_4)(\eta+2)\begin{bmatrix} \nabla^2 w_{0,xx} \\ \nabla^2 w_{0,yy} \\ \nabla^2 w_{0,xy} \end{bmatrix}, \\ \frac{1}{\mu}\begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} &= \bar{\epsilon}^2\{2(\tfrac{1}{2}\bar{z}^2 + B_1\bar{z})(\eta+1) + \eta B_2 + (\eta+2)B_3\}\begin{bmatrix} \nabla^2 w_{0,x} \\ \nabla^2 w_{0,y} \end{bmatrix}, \\ \sigma_{zz} &= 0, \end{aligned}\right\} \tag{2.57}$$

where

$$\nabla^4 w_0 = 0. \tag{2.58}$$

If again we require the lateral surface tractions to be zero, as in generalized plane stress, then (2.57) gives that

$$B_1 = 0, \quad \eta + 1 + \eta B_2 + (\eta+2) B_3 = 0.$$

The further requirement that the displacement of the mid-plane coincides with that given by classical thin plate theory then imposes the conditions that

$$B_2 = 0, \quad B_4 = 0,$$

thereby recovering the exact solution given by Love (1927, article 304) for plane stress bending of moderately thick plates.

We note that in this case also an exact three-dimensional solution (2.56)–(2.57) is generated by any solution of a two-dimensional equation, namely the biharmonic equation (2.58), which is the equation of classical thin plate theory for bending by edge couples.

Again, the solution (2.57) does not allow pointwise specification of the edge displacement or traction, but it is possible to specify at the edge of the plate the value of the mid-surface displacement $w(x,y,0)$ and either its normal derivative $n_x w(x,y,0)_{,x} + n_y w(x,y,0)_{,y}$ or the edge bending moment $n_x M_{xx} + n_y M_{xy}$.

3. Classical laminate theory

(a) *Laminate geometry*

We now consider laminated plates comprised of $2N+1$ homogeneous laminae, each of which is of an isotropic linearly elastic material. In general, the laminae are of different materials, with differing elastic constants. For convenience, we confine our attention to laminates which are symmetric about the mid-plane $Z = 0$, so that the ith lamina above the mid-plane is identical in material and thickness to the ith lamina below the mid-plane. For such a symmetric laminate the stretching and bending solutions uncouple. The procedure we shall follow can be extended to include asymmetric laminates, but in this case the stretching and bending deformations cannot be separated, and the algebraic complexity of the analysis is increased.

Any quantity related to the ith lamina will be identified by the index i. The layers are numbered according to the scheme shown in figure 1, where the layer $i = 0$ contains the mid-

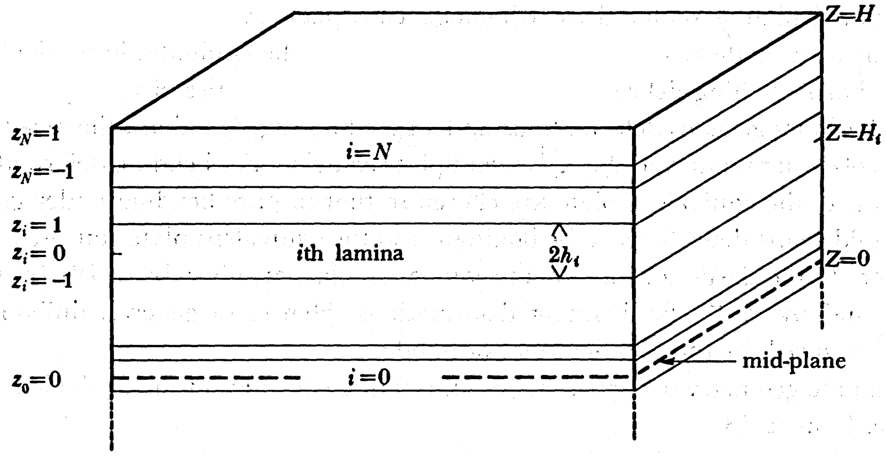

Figure 1. Laminate geometry and notation for symmetric lay-up about $Z = 0$.

plane of the laminated plate and the layer $i = N$ is the layer adjacent to the upper surface. The ith layer has uniform thickness $2h_i$ and Lamé elastic constants λ_i and μ_i. We also use the elastic constants λ_i' and η_i, defined as

$$\lambda_i' = 2\lambda_i\mu_i/(\lambda_i+2\mu_i), \quad \eta_i = \lambda_i'/2\mu_i. \tag{3.1}$$

The overall laminate thickness is denoted by $2H$, so that

$$H = h_0 + 2\sum_{i=1}^{N} h_i. \tag{3.2}$$

Also, we denote by H_i the distance from the mid-plane of the plate to the mid-plane of lamina i, so that

$$H_0 = 0, \quad H_1 = h_0 + h_1, \quad H_i = h_0 + 2\sum_{r=1}^{i-1} h_r + h_i \quad (i = 2, 3, \ldots, N). \tag{3.3}$$

In particular, $H_N = H - h_N$.

We use scaled variables x, y, z, u, v and w as in (2.12), where now H is the overall laminate half-thickness, and $Z = 0$ coincides with the mid-plane of the laminated plate. In addition we introduce scaled local coordinates z_i in the z-direction in each lamina, defined by

$$z_i = Z_i/h_i, \quad Z_i = Z - H_i \quad (i = 0, 1, 2, \ldots, N). \tag{3.4}$$

Thus $z_i = 0$ is the mid-plane of layer i, and $z_i = \pm 1$ are the upper and lower surfaces of layer i. In addition we use the parameter ϵ defined in (2.13) and also denote

$$\epsilon_i = h_i/a \quad (i = 0, 1, \ldots, N). \tag{3.5}$$

It then follows from (3.2) that

$$\epsilon = \epsilon_0 + 2\sum_{i=1}^{N} \epsilon_i. \tag{3.6}$$

(b) *The equivalent plate*

In classical laminate theory, as described by, for example, Christensen (1979), the inhomogeneous laminated plate is replaced by a homogeneous plate which we term the equivalent plate. The equivalent plate has the same overall geometry as the laminate; in particular, it has thickness $2H$. For stretching deformations, the mechanical properties of the equivalent plate are such that for homogeneous deformations under specified edge tractions, the mean in-plane displacements of the laminate and the equivalent plate coincide. For bending deformations, the elastic constants of the equivalent plate are chosen so that in pure bending under given edge moment, the mid-plane deflections of the laminate and the equivalent plate coincide. Thus, for isotropic materials, the elastic constants of the equivalent plate are given by (2.18) for stretching deformations and by (2.27) for bending deformations. Hence, in general, different elastic constants are required for the two deformation modes.

For the laminate geometry described in §3*a*, the elastic constants of the equivalent plate for stretching are, from (2.18),

$$\begin{bmatrix} \hat{\lambda}' \\ \hat{\mu} \end{bmatrix} = \frac{1}{H}\left\{h_0 \begin{bmatrix} \lambda'_0 \\ \mu_0 \end{bmatrix} + 2\sum_{i=1}^{N} h_i \begin{bmatrix} \lambda'_i \\ \mu_i \end{bmatrix}\right\} = \frac{1}{\epsilon}\left\{\epsilon_0 \begin{bmatrix} \lambda'_0 \\ \mu_0 \end{bmatrix} + 2\sum_{i=1}^{N} \epsilon_i \begin{bmatrix} \lambda'_i \\ \mu_i \end{bmatrix}\right\}. \tag{3.7}$$

We also define

$$\hat{\eta} = \hat{\lambda}'/2\hat{\mu}. \tag{3.8}$$

In general, the quantities associated with stretching deformations of the equivalent plate will be denoted by superposed circumflexes. Thus the through-thickness average in-plane displacement $(\hat{u}, \hat{v})$ (scaled as in (2.12)) of the equivalent plate is governed by the equations

$$2(\hat{\eta}+1)\,\hat{\Delta}_{,x} - \hat{\Omega}_{,y} = 0, \quad 2(\hat{\eta}+1)\,\hat{\Delta}_{,y} + \hat{\Omega}_{,x} = 0, \tag{3.9}$$

$$\hat{\Delta} = \hat{u}_{,x} + \hat{v}_{,y}, \quad \hat{\Omega} = \hat{v}_{,x} - \hat{u}_{,y}, \tag{3.10}$$

where $\hat{\Delta}$ and $\hat{\Omega}$ are harmonic functions. Similarly the in-plane stress resultants for the equivalent plate are

$$\begin{bmatrix} \hat{N}_{xx} \\ \hat{N}_{yy} \\ \hat{N}_{xy} \end{bmatrix} = 2H\hat{\mu} \begin{bmatrix} 2(\hat{\eta}+1) & 2\hat{\eta} & 0 \\ 2\hat{\eta} & 2(\hat{\eta}+1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_{,x} \\ \hat{v}_{,y} \\ \hat{u}_{,y} + \hat{v}_{,x} \end{bmatrix}. \tag{3.11}$$

Quantities associated with bending deformations of the equivalent plate will be denoted by superposed tildes. From (2.27), the elastic constants for bending of the equivalent plate are

$$\begin{bmatrix}\tilde{\lambda}'\\ \tilde{\mu}\end{bmatrix} = \frac{1}{H^3}\left\{h_0^3\begin{bmatrix}\lambda_0'\\ \mu_0\end{bmatrix} + \sum_{i=1}^{N}[(H_i+h_i)^3-(H_i-h_i)^3]\begin{bmatrix}\lambda_i'\\ \mu_i\end{bmatrix}\right\}, \tag{3.12}$$

and we also define

$$\tilde{\eta} = \tilde{\lambda}'/2\tilde{\mu}. \tag{3.13}$$

The deflection $\tilde{w}$ (scaled as in (2.12)) of the mid-surface of the equivalent plate is governed by the biharmonic equation

$$\nabla^4\tilde{w} = 0, \tag{3.14}$$

and the bending moments in the equivalent plate are given by

$$\begin{bmatrix}\tilde{M}_{xx}\\ \tilde{M}_{yy}\\ \tilde{M}_{xy}\end{bmatrix} = -\tfrac{2}{3}\epsilon\tilde{\mu}H^2\begin{bmatrix}2(\tilde{\eta}+1) & 2\tilde{\eta} & 0\\ 2\tilde{\eta} & 2(\tilde{\eta}+1) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{w}_{,xx}\\ \tilde{w}_{,yy}\\ 2\tilde{w}_{,xy}\end{bmatrix}. \tag{3.15}$$

4. Exact theory for laminates

We now present a theory in which the three-dimensional field equations and interface conditions are exactly satisfied. The only limitation is that in this theory (as in all other plate theories) edge boundary conditions can be satisfied only in an average fashion, rather than point by point.

(a) *Stretching deformations*

We first consider the stress and deformation in stretching deformations of the laminate. For these, we seek solutions in which, in each lamina, the displacement is of the form (2.51), with associated stress of the form (2.52), and with $\bar{\epsilon}$, $\bar{z}$, η and μ replaced by ϵ_i, z_i, η_i and μ_i respectively. The constants S_1, S_2 and S_3 are in general different in each lamina. Their values in the ith lamina are denoted by $S_1^{(i)}$, $S_2^{(i)}$ and $S_3^{(i)}$.

The functions $u_0(x,y)$ and $v_0(x,y)$ are at our disposal, subject only to their satisfying (2.54). Because classical laminate theory appears to give satisfactory results for the average values of u and v, we choose $u_0(x,y)$ and $v_0(x,y)$ to be the same in each layer, and to be the displacement components $\hat{u}(x,y)$ and $\hat{v}(x,y)$, under given boundary conditions, of the equivalent plate, and so they are solutions of (3.9) and (3.10). Thus, in layer i, the displacement and stress are assumed to have the forms

$$\begin{bmatrix}u^{(i)}\\ v^{(i)}\end{bmatrix} = \begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix} - \epsilon_i^2(\tfrac{1}{2}z_i^2 + S_2^{(i)}z_i + S_3^{(i)})\left\{(\eta_i+2)\begin{bmatrix}\hat{\Delta}_{,x}\\ \hat{\Delta}_{,y}\end{bmatrix} + \begin{bmatrix}-\hat{\Omega}_{,y}\\ \hat{\Omega}_{,x}\end{bmatrix}\right\}, \tag{4.1}$$

$$w^{(i)} = -\epsilon_i\eta_i(z_i+S_1^{(i)})\hat{\Delta}, \tag{4.2}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma_{xx}^{(i)}\\ \sigma_{yy}^{(i)}\\ \sigma_{xy}^{(i)}\end{bmatrix} = \begin{bmatrix}2\hat{u}_{,x}\\ 2\hat{v}_{,y}\\ \hat{u}_{,y}+\hat{v}_{,x}\end{bmatrix} + 2\eta_i\begin{bmatrix}\hat{\Delta}\\ \hat{\Delta}\\ 0\end{bmatrix} - \epsilon_i^2(\tfrac{1}{2}z_i^2 + S_2^{(i)}z_i + S_3^{(i)})\left\{2(\eta_i+2)\begin{bmatrix}\hat{\Delta}_{,xx}\\ \hat{\Delta}_{,yy}\\ \hat{\Delta}_{,xy}\end{bmatrix} + \begin{bmatrix}-2\hat{\Omega}_{,xy}\\ 2\hat{\Omega}_{,xy}\\ \hat{\Omega}_{,xx}-\hat{\Omega}_{,yy}\end{bmatrix}\right\}, \tag{4.3}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma_{xz}^{(i)}\\ \sigma_{yz}^{(i)}\end{bmatrix} = -\epsilon_i\{2(\eta_i+1)z_i + S_1^{(i)}\eta_i + S_2^{(i)}(\eta_i+2)\}\begin{bmatrix}\hat{\Delta}_{,x}\\ \hat{\Delta}_{,y}\end{bmatrix} - \epsilon_i(z_i+S_2^{(i)})\begin{bmatrix}-\hat{\Omega}_{,y}\\ \hat{\Omega}_{,x}\end{bmatrix}, \tag{4.4}$$

$$\sigma_{zz}^{(i)} = 0. \tag{4.5}$$

Here $\hat{\Delta}$ and $\hat{\Omega}$ satisfy (3.9), are related to $\hat{u}$ and $\hat{v}$ by (3.10) and are functions of x and y only.

By using (3.9) and (3.10), (4.1), (4.3) and (4.4) can be written as

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} - \epsilon_i^2(\tfrac{1}{2}z_i^2 + S_2^{(i)} z_i + S_3^{(i)})(\eta_i - 2\hat{\eta}) \begin{bmatrix} \hat{\Delta}_{,x} \\ \hat{\Delta}_{,y} \end{bmatrix}, \tag{4.6}$$

$$\frac{1}{\mu_i}\begin{bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \sigma_{xy}^{(i)} \end{bmatrix} = \frac{1}{\hat{\mu}}\begin{bmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{xy} \end{bmatrix} + \frac{\eta_i - \hat{\eta}}{\hat{\mu}(1+2\hat{\eta})}\begin{bmatrix} \hat{\sigma}_{xx} + \hat{\sigma}_{yy} \\ \hat{\sigma}_{xx} + \hat{\sigma}_{yy} \\ 0 \end{bmatrix} - 2\epsilon_i^2(\tfrac{1}{2}z_i^2 + S_2^{(i)} z_i + S_3^{(i)})(\eta_i - 2\hat{\eta})\begin{bmatrix} \hat{\Delta}_{,xx} \\ \hat{\Delta}_{,yy} \\ \hat{\Delta}_{,xy} \end{bmatrix}, \tag{4.7}$$

$$\frac{1}{\mu_i}\begin{bmatrix} \sigma_{xz}^{(i)} \\ \sigma_{yz}^{(i)} \end{bmatrix} = -\epsilon_i\{2(\eta_i - \hat{\eta}) z_i + S_1^{(i)}\eta_i + S_2^{(i)}(\eta_i - 2\hat{\eta})\}\begin{bmatrix} \hat{\Delta}_{,x} \\ \hat{\Delta}_{,y} \end{bmatrix}, \tag{4.8}$$

where $\hat{\sigma}_{xx}$, $\hat{\sigma}_{yy}$ and $\hat{\sigma}_{xy}$ denote the stress associated with the equivalent plate. These expressions represent an exact solution in each layer provided that $\hat{u}$ and $\hat{v}$ satisfy (3.9) and (3.10).

For stretching solutions of symmetric laminates u and v are even functions of z, and w is an odd function of z, so it is sufficient to consider the region $z \geqslant 0$. This involves the layers labelled $i = 0, 1, \ldots, N$. Hence we have at our disposal $3N+3$ constants $S_\alpha^{(i)}$ ($\alpha = 1, 2, 3$; $i = 0, 1, \ldots, N$).

The following conditions have to be satisfied:

(i) Symmetry conditions at $z_0 = 0$:

$$w^{(0)} = 0, \quad \sigma_{xz}^{(0)} = 0, \quad \sigma_{yz}^{(0)} = 0, \quad \text{at} \quad z = 0. \tag{4.9}$$

(ii) Continuity of displacement and traction at each interface between layer $i-1$ at $z_{i-1} = 1$ and layer i at $z_i = -1$ ($i = 1, 2, \ldots, N$):

$$u^{(i-1)} = u^{(i)}, \quad v^{(i-1)} = v^{(i)}, \quad w^{(i-1)} = w^{(i)}, \tag{4.10}$$

$$\sigma_{xz}^{(i-1)} = \sigma_{xz}^{(i)}, \quad \sigma_{yz}^{(i-1)} = \sigma_{yz}^{(i)}. \tag{4.11}$$

No continuity conditions on σ_{zz} are required, because $\sigma_{zz} = 0$ throughout the laminate. We leave aside for now the traction condition on the upper surface $z = 1$.

Because of the form of the solution (4.1)–(4.4), the above conditions all reduce to conditions on the constants $S_\alpha^{(i)}$. The number of the conditions (4.9)–(4.11) exceeds the number of available constants. However, the conditions are not all independent. From (4.8) it is clear that the conditions for $\sigma_{xz} = 0$ and $\sigma_{yz} = 0$ at $z = 0$ are identical, and that the conditions for σ_{xz} and σ_{yz} to be continuous at the interface between layers $i-1$ and i are also identical. Furthermore, from (4.6), the conditions for continuity of u and v at the interface between layers $i-1$ and i are identical too. Thus (4.9)–(4.11) yield $3N+2$ independent conditions on the $3N+3$ constants $S_\alpha^{(i)}$.

From (4.2) and (4.8), the conditions (4.9) are satisfied if

$$S_1^{(0)} = 0, \quad S_2^{(0)} = 0. \tag{4.12}$$

At the interface between layers $i-1$ and i, continuity of w gives, from (4.2),

$$\epsilon_{i-1}\eta_{i-1}(S_1^{(i-1)} + 1) = \epsilon_i \eta_i (S_1^{(i)} - 1), \tag{4.13}$$

and continuity of σ_{xz} and σ_{yz} gives, from (4.8),

$$\mu_{i-1}\epsilon_{i-1}\{\eta_{i-1}S_1^{(i-1)}+(\eta_{i-1}-2\hat{\eta})S_2^{(i-1)}+2(\eta_{i-1}-\hat{\eta})\} = \mu_i\epsilon_i\{\eta_i S_1^{(i)}+(\eta_i-2\hat{\eta})S_2^{(i)}-2(\eta_i-\hat{\eta})\}, \tag{4.14}$$

whereas continuity of u and v gives, from (4.6),

$$\epsilon_{i-1}^2(S_3^{(i-1)}+\tfrac{1}{2}+S_2^{(i-1)})(\eta_{i-1}-2\hat{\eta}) = \epsilon_i^2(S_3^{(i)}+\tfrac{1}{2}-S_2^{(i)})(\eta_i-2\hat{\eta}). \tag{4.15}$$

Each of (4.13), (4.14) and (4.15) is a recurrence relation of the form

$$a_{i-1}+b_{i-1} = a_i-b_i \quad (i = 1,2,\ldots,N), \tag{4.16}$$

where, in (4.13), $$a_i = \epsilon_i\eta_i S_1^{(i)}, \quad b_i = \epsilon_i\eta_i; \tag{4.17}$$

in (4.14) $$a_i = \mu_i\epsilon_i\{\eta_i S_1^{(i)}+(\eta_i-2\hat{\eta})S_2^{(i)}\}, \quad b_i = 2\mu_i\epsilon_i(\eta_i-\hat{\eta}); \tag{4.18}$$

and in (4.15) $$a_i = \epsilon_i^2(S_3^{(i)}+\tfrac{1}{2})(\eta_i-2\hat{\eta}), \quad b_i = \epsilon_i^2 S_2^{(i)}(\eta_i-2\hat{\eta}). \tag{4.19}$$

By replacing i by r in (4.16), and summing the resulting relations from $r = 1$ to $r = i$, we obtain

$$a_i = a_0+\sum_{r=1}^{i}(b_{r-1}+b_r) = a_0+b_0+2\sum_{r=1}^{i-1}b_r+b_i. \tag{4.20}$$

Application of this solution in turn to (4.17), (4.18) and (4.19), and the use of (4.12) gives

$$\epsilon_i\eta_i S_1^{(i)} = \sum_{r=1}^{i}(\epsilon_{r-1}\eta_{r-1}+\epsilon_r\eta_r), \tag{4.21}$$

$$\mu_i\epsilon_i\{\eta_i S_1^{(i)}+(\eta_i-2\hat{\eta})S_2^{(i)}\} = 2\sum_{r=1}^{i}\{\mu_{r-1}\epsilon_{r-1}(\eta_{r-1}-\hat{\eta})+\mu_r\epsilon_r(\eta_r-\hat{\eta})\}, \tag{4.22}$$

$$\epsilon_i^2(S_3^{(i)}+\tfrac{1}{2})(\eta_i-2\hat{\eta}) = \epsilon_0^2(S_3^{(0)}+\tfrac{1}{2})(\eta_0-2\hat{\eta})+\sum_{r=1}^{i}\{\epsilon_{r-1}^2 S_2^{(r-1)}(\eta_{r-1}-2\hat{\eta})+\epsilon_r^2 S_2^{(r)}(\eta_r-2\hat{\eta})\}, \tag{4.23}$$

all for $i = 1,2,\ldots,N$.

Equations (4.21) and (4.22) determine $S_1^{(i)}$ and $S_2^{(i)}$ explicitly. Equation (4.23) then determines $S_3^{(i)}$ $(i = 1,2,\ldots,N)$ in terms of $S_3^{(0)}$. Thus $S_3^{(0)}$ remains the only undetermined constant.

We now consider the shear traction on the upper surface $z = 1$ (or $z_N = 1$), which is given by (4.8) with $i = N$. From (4.22), with $i = N$, and using (3.7) and (3.8), we have

$$\begin{aligned}\mu_N\epsilon_N\{2(\eta_N-\hat{\eta})+\eta_N S_1^{(N)}+(\eta_N-2\hat{\eta})S_2^{(N)}\}\\ &= 2\mu_N\epsilon_N(\eta_N-\hat{\eta})+2\sum_{r=1}^{N}\{\mu_{r-1}\epsilon_{r-1}(\eta_{r-1}-\hat{\eta})+\mu_r\epsilon_r(\eta_r-\hat{\eta})\}\\ &= 2\mu_0\epsilon_0(\eta_0-\hat{\eta})+4\sum_{r=1}^{N}\mu_r\epsilon_r(\eta_r-\hat{\eta})\\ &= 2\left\{\mu_0\epsilon_0\eta_0+2\sum_{r=1}^{N}\mu_r\epsilon_r\eta_r\right\}-2\hat{\eta}\left\{\mu_0\epsilon_0+2\sum_{r=1}^{N}\mu_r\epsilon_r\right\}\\ &= 2(\hat{\mu}\epsilon\hat{\eta}-\hat{\eta}\epsilon\hat{\mu}) = 0.\end{aligned}$$

Hence, from (4.8) (with $i = N$), the expressions (4.22) imply that

$$\sigma_{xz} = 0, \quad \sigma_{yz} = 0, \tag{4.24}$$

at the lateral surface $z = 1$. Thus it follows that the lateral surfaces of the laminate are necessarily traction-free, and it is neither necessary nor possible to impose additional boundary conditions on these surfaces.

To assign the remaining undetermined constant $S_3^{(0)}$, we may impose one further condition. A possible choice is to require that the mean in-plane displacement components $\bar{u}(x,y)$ and $\bar{v}(x,y)$ coincide with the mean displacement components $\hat{u}(x,y)$ and $\hat{v}(x,y)$ for the equivalent plate. From (4.6) by integrating through the plate thickness from $z = -1$ to $z = 1$, we have

$$\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} - \frac{1}{\epsilon}\left\{\epsilon_0^3(\tfrac{1}{6}+S_3^{(0)})(\eta_0-2\hat{\eta}) + 2\sum_{i=1}^{N}\epsilon_i^3(\tfrac{1}{6}+S_3^{(i)})(\eta_i-2\hat{\eta})\right\}\begin{bmatrix} \hat{\Delta}_{,x} \\ \hat{\Delta}_{,y} \end{bmatrix}. \tag{4.25}$$

Hence $(\bar{u},\bar{v}) = (\hat{u},\hat{v})$ if we choose

$$\epsilon_0^3(\tfrac{1}{6}+S_3^{(0)})(\eta_0-2\hat{\eta}) + 2\sum_{i=1}^{N}\epsilon_i^3(\tfrac{1}{6}+S_3^{(i)})(\eta_i-2\hat{\eta}) = 0. \tag{4.26}$$

From (4.23), it then follows that $S_3^{(0)}$ can be expressed in terms of the previously determined constants $S_2^{(i)}$ by the relation

$$\epsilon\epsilon_0^2(\tfrac{1}{2}+S_3^{(0)})(\eta_0-2\hat{\eta}) = \tfrac{1}{3}\left\{\epsilon_0^3(\eta_0-2\hat{\eta}) + 2\sum_{r=1}^{N}\epsilon_r^3(\eta_r-2\hat{\eta})\right\}$$
$$-2\sum_{r=1}^{N}\sum_{i=1}^{r}\epsilon_r\{\epsilon_{i-1}^2 S_2^{(i-1)}(\eta_{i-1}-2\hat{\eta}) + \epsilon_i^2 S_2^{(i)}(\eta_i-2\hat{\eta})\}. \tag{4.27}$$

This is the natural choice of the condition to determine $S_3^{(0)}$ when boundary conditions on the mean in-plane displacement are prescribed at the edge of the plate.

Alternatively, we may require that the stress resultants N_{xx}, N_{yy}, N_{xy} coincide with the corresponding stress resultants $\hat{N}_{xx}$, $\hat{N}_{yy}$, $\hat{N}_{xy}$ for the equivalent plate. Equations (2.16) and (4.7) show that

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \begin{bmatrix} \hat{N}_{xx} \\ \hat{N}_{yy} \\ \hat{N}_{xy} \end{bmatrix} - \frac{4H}{\epsilon}\left\{\epsilon_0^3\mu_0(\tfrac{1}{6}+S_3^{(0)})(\eta_0-2\hat{\eta}) + 2\sum_{i=1}^{N}\epsilon_i^3\mu_i(\tfrac{1}{6}+S_3^{(i)})(\eta_i-2\hat{\eta})\right\}\begin{bmatrix} \hat{\Delta}_{,xx} \\ \hat{\Delta}_{,yy} \\ \hat{\Delta}_{,xy} \end{bmatrix}. \tag{4.28}$$

Hence $(N_{xx}, N_{yy}, N_{xy}) = (\hat{N}_{xx}, \hat{N}_{yy}, \hat{N}_{xy})$ if

$$\epsilon_0^3\mu_0(\tfrac{1}{6}+S_3^{(0)})(\eta_0-2\hat{\eta}) + 2\sum_{i=1}^{N}\epsilon_i^3\mu_i(\tfrac{1}{6}+S_3^{(i)})(\eta_i-2\hat{\eta}) = 0. \tag{4.29}$$

From (3.7) and (4.23) it follows that in this case $S_3^{(0)}$ is given in terms of the previously determined constants $S_2^{(i)}$ by the relation

$$\epsilon\epsilon_0^2\hat{\mu}(\tfrac{1}{2}+S_3^{(0)})(\eta_0-2\hat{\eta}) = \tfrac{1}{3}\left\{\epsilon_0^3\mu_0(\eta_0-2\hat{\eta}) + 2\sum_{i=1}^{N}\epsilon_i^3\mu_i(\eta_i-2\hat{\eta})\right\}$$
$$-2\sum_{r=1}^{N}\sum_{i=1}^{r}\epsilon_r\mu_r\{\epsilon_{i-1}^2 S_2^{(i-1)}(\eta_{i-1}-2\hat{\eta}) + \epsilon_i^2 S_2^{(i)}(\eta_i-2\hat{\eta})\}. \tag{4.30}$$

This is the natural way to specify $S_3^{(0)}$ if traction boundary conditions at the edge of the plate are prescribed.

We observe that the two conditions (4.27) and (4.30) coincide only when the shear moduli μ_i are all the same. We note that in (4.22)–(4.30) the constants $S_2^{(i)}$ and $S_3^{(i)}$ occur only in conjunction with the factors $(\eta_i - 2\hat{\eta})$. Thus the numerical evaluation of the constants is simplified by introducing new constants $T_2^{(i)}$, $T_3^{(i)}$, defined as

$$T_2^{(i)} = (\eta_i - 2\hat{\eta})\, S_2^{(i)}, \quad T_3^{(i)} = (\eta_i - 2\hat{\eta})\, S_3^{(i)}.$$

When $S_1^{(i)}$, $S_2^{(i)}$ and $S_3^{(i)}$ are determined by (4.12), (4.21)–(4.23) and either (4.27) or (4.30), then equations (4.2) and (4.5)–(4.8) comprise exact three-dimensional solutions for stretching deformations of the heterogeneous laminated plate, with zero tractions on the lateral surfaces. To construct explicit solutions of this form, it is necessary to solve only the two-dimensional equations (3.9) and (3.10) for the homogeneous equivalent plate. We note that the constants $S_1^{(i)}$, $S_2^{(i)}$ and $S_3^{(i)}$ depend only on the thicknesses and elastic constants of the laminae, and not on any particular boundary-value problem considered.

A quantity of interest, about which classical laminate theory gives no information, is the interfacial shear stress between laminae. If we denote by $\boldsymbol{\tau}_i$ the shear traction between layers $i-1$ and i, then $\boldsymbol{\tau}_i$ has x and y components $\sigma_{xz}^{(i)}$ and $\sigma_{yz}^{(i)}$, evaluated at $z_i = -1$. Hence, from (4.8),

$$\boldsymbol{\tau}_i = -\mu_i \epsilon_i \{-2(\eta_i - \hat{\eta}) + S_1^{(i)}\eta_i + S_2^{(i)}(\eta_i - 2\hat{\eta})\}\, \mathrm{grad}\, \hat{\Delta}. \tag{4.31}$$

Therefore, from (4.22),

$$\boldsymbol{\tau}_i = -2\{\mu_0 \epsilon_0 (\eta_0 - \hat{\eta}) + 2\sum_{r=1}^{i-1} \epsilon_r \mu_r (\eta_r - \hat{\eta})\}\, \mathrm{grad}\, \hat{\Delta}. \tag{4.32}$$

If $\eta_i = \hat{\eta}$ in each layer (that is, if the material in each layer has the same value of Poisson's ratio), then $\boldsymbol{\tau}_i = 0$ at each interface. It also follows from (4.32) that the jump in the shear traction from the lower surface to the upper surface of layer i is

$$\boldsymbol{\tau}_{i+1} - \boldsymbol{\tau}_i = -4\mu_i \epsilon_i (\eta_i - \hat{\eta})\, \mathrm{grad}\, \hat{\Delta}.$$

The sign of this jump is governed by the sign of $\eta_i - \hat{\eta}$. Thus the magnitude of $\boldsymbol{\tau}_i$ can be strongly influenced by the choice of stacking sequence for the laminate.

(b) *Bending deformations*

For bending deformations of the laminate we seek solutions in which, in each lamina, the displacement is of the form (2.56) and the associated stress is of the form (2.57), with $\bar{\epsilon}$, $\bar{z}$, η and μ replaced by ϵ_i, z_i, η_i and μ_i respectively. The values of the constants B_1, B_2, B_3 and B_4 for the ith layer are denoted by $B_1^{(i)}$, $B_2^{(i)}$, $B_3^{(i)}$ and $B_4^{(i)}$ respectively. The function $w_0(x, y)$ is chosen to be the displacement $\tilde{w}(x, y)$, under suitable boundary conditions, of the equivalent plate, and so satisfies the biharmonic equation (3.14). Thus in layer i the displacement and stress have the forms

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \end{bmatrix} = -\epsilon_i (z_i + B_1^{(i)}) \begin{bmatrix} \tilde{w}_{,x} \\ \tilde{w}_{,y} \end{bmatrix} + \epsilon_i^3 (\tfrac{1}{6} z_i^3 + \tfrac{1}{2} B_1^{(i)} z_i^2 + B_3^{(i)} z_i + B_4^{(i)})\,(\eta_i + 2) \begin{bmatrix} \nabla^2 \tilde{w}_{,x} \\ \nabla^2 \tilde{w}_{,y} \end{bmatrix}, \tag{4.33}$$

$$w^{(i)} = \tilde{w} + \epsilon_i^2 \eta_i (\tfrac{1}{2} z_i^2 + B_1^{(i)} z_i + B_2^{(i)})\, \nabla^2 \tilde{w}, \tag{4.34}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma^{(i)}_{xx}\\ \sigma^{(i)}_{yy}\\ \sigma^{(i)}_{xy}\end{bmatrix} = -2\epsilon_i(z_i+B^{(i)}_1)\left\{\eta_i\begin{bmatrix}\nabla^2\tilde{w}\\ \nabla^2\tilde{w}\\ 0\end{bmatrix}+\begin{bmatrix}\tilde{w}_{,xx}\\ \tilde{w}_{,yy}\\ \tilde{w}_{,xy}\end{bmatrix}\right\}$$

$$+2\epsilon^3_i(\tfrac{1}{6}z^3_i+\tfrac{1}{2}B^{(i)}_1z^2_i+B^{(i)}_3z_i+B^{(i)}_4)(\eta_i+2)\begin{bmatrix}\nabla^2\tilde{w}_{,xx}\\ \nabla^2\tilde{w}_{,yy}\\ \nabla^2\tilde{w}_{,xy}\end{bmatrix}, \tag{4.35}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma^{(i)}_{xz}\\ \sigma^{(i)}_{yz}\end{bmatrix} = \epsilon^2_i\{2(\tfrac{1}{2}z^2_i+B^{(i)}_1z_i)(\eta_i+1)+\eta_iB^{(i)}_2+(\eta_i+2)B^{(i)}_3\}\begin{bmatrix}\nabla^2\tilde{w}_{,x}\\ \nabla^2\tilde{w}_{,y}\end{bmatrix}, \tag{4.36}$$

$$\sigma^{(i)}_{zz} = 0. \tag{4.37}$$

As before, for symmetric laminates it is sufficient to consider the region $z \geqslant 0$; for bending deformations, w is an even function of z and u and v are odd functions of z. Hence we require conditions to determine $4(N+1)$ constants $B^{(i)}_\alpha$ ($\alpha = 1, 2, 3, 4$; $i = 0, 1, 2, \ldots, N$). The available conditions are:

(i) Symmetry conditions at $z = 0$:

$$u^{(0)} = 0, \quad v^{(0)} = 0, \quad \sigma^{(0)}_{xx} = 0, \quad \sigma^{(0)}_{yy} = 0, \quad \sigma^{(0)}_{xy} = 0, \quad \text{at } z_0 = 0.$$

(ii) Continuity of displacement and traction at each interface between layer $i-1$ at $z_{i-1} = 1$ and layer i at $z_i = -1$ ($i = 1, 2, \ldots, N$):

$$\left.\begin{aligned} u^{(i-1)} &= u^{(i)}, & v^{(i-1)} &= v^{(i)}, & w^{(i-1)} &= w^{(i)},\\ \sigma^{(i-1)}_{yz} &= \sigma^{(i)}_{yz}, & \sigma^{(i-1)}_{xz} &= \sigma^{(i)}_{xz}, & \sigma^{(i-1)}_{zz} &= \sigma^{(i)}_{zz}.\end{aligned}\right\} \tag{4.38}$$

(iii) Zero traction on the lateral boundary of the laminate:

$$\sigma^{(N)}_{yz} = 0, \quad \sigma^{(N)}_{xz} = 0, \quad \sigma^{(N)}_{zz} = 0, \quad \text{at } z_N = 1. \tag{4.39}$$

As for stretching deformations, the conditions are not all independent. Inspection of (4.33)–(4.36) shows that (i)–(iii) imply just $4N+3$ independent conditions on the constants $B^{(i)}_\alpha$, leaving us once again with one disposable constant.

From (4.33) and (4.35), conditions (i) are satisfied if

$$B^{(0)}_1 = 0, \quad B^{(0)}_4 = 0. \tag{4.40}$$

Continuity of u and v at the interfaces is assured if, from (4.33),

$$\epsilon_{i-1}(1+B^{(i-1)}_1) = \epsilon_i(-1+B^{(i)}_1), \tag{4.41}$$

$$\epsilon^3_{i-1}(\tfrac{1}{6}+\tfrac{1}{2}B^{(i-1)}_1+B^{(i-1)}_3+B^{(i-1)}_4)(\eta_{i-1}+2) = \epsilon^3_i(-\tfrac{1}{6}+\tfrac{1}{2}B^{(i)}_1-B^{(i)}_3+B^{(i)}_4)(\eta_i+2). \tag{4.42}$$

From (4.34), continuity of w requires that

$$\epsilon^2_{i-1}\eta_{i-1}(\tfrac{1}{2}+B^{(i-1)}_1+B^{(i-1)}_2) = \epsilon^2_i\eta_i(\tfrac{1}{2}-B^{(i)}_1+B^{(i)}_2), \tag{4.43}$$

and continuity of the interfacial shear stress gives, from (4.36),

$$\mu_{i-1}\epsilon^2_{i-1}\{2(\tfrac{1}{2}+B^{(i-1)}_1)(\eta_{i-1}+1)+\eta_{i-1}B^{(i-1)}_2+(\eta_{i-1}+2)B^{(i-1)}_3\}$$
$$= \mu_i\epsilon^2_i\{2(\tfrac{1}{2}-B^{(i)}_1)(\eta_i+1)+\eta_iB^{(i)}_2+(\eta_i+2)B^{(i)}_3\}. \tag{4.44}$$

The condition (iii) gives

$$2(\tfrac{1}{2}+B^{(N)}_1)(\eta_N+1)+\eta_NB^{(N)}_2+(\eta_N+2)B^{(N)}_3 = 0. \tag{4.45}$$

Each of equations (4.41)–(4.44) is a recurrence relation of the form (4.16) with solution of the form (4.20). The solution of $(4.40)_1$ and (4.41) determines the $B_1^{(i)}$ as

$$\epsilon_i B_1^{(i)} = \sum_{r=1}^{i} (\epsilon_{r-1}+\epsilon_r) = \epsilon_0+\epsilon_i+2\sum_{r=1}^{i-1}\epsilon_i.$$

Hence, from (3.3) and (3.5), $$h_i B_1^{(i)} = H_i. \tag{4.46}$$

By solving the recurrence relation (4.43), we obtain

$$\epsilon_i^2\eta_i(\tfrac{1}{2}+B_2^{(i)}) = \epsilon_0^2\eta_0(\tfrac{1}{2}+B_2^{(0)})+\sum_{r=1}^{i}(\epsilon_{r-1}^2\eta_{r-1}B_1^{(r-1)}+\epsilon_r^2\eta_r B_1^{(r)}), \tag{4.47}$$

which determines $B_2^{(i)}$ $(i = 1, 2, \ldots, N)$ in terms of $B_2^{(0)}$ and the already evaluated constants $B_1^{(i)}$. Similarly, from (4.44), the $B_3^{(i)}$ are given in terms of $B_2^{(0)}$, $B_3^{(0)}$, $B_1^{(i)}$ and $B_2^{(i)}$ by

$$\mu_i\epsilon_i^2\{(\eta_i+1)+\eta_i B_2^{(i)}+(\eta_i+2)B_3^{(i)}\} = \mu_0\epsilon_0^2\{(\eta_0+1)+\eta_0 B_2^{(0)}+(\eta_0+2)B_3^{(0)}\}$$
$$+2\sum_{r=1}^{i}\{\mu_{r-1}\epsilon_{r-1}^2(\eta_{r-1}+1)B_1^{(r-1)}+\mu_r\epsilon_r^2(\eta_r+1)B_1^{(r)}\}. \tag{4.48}$$

By solving (4.42) and using (4.40), we determine $B_4^{(i)}$ in terms of $B_3^{(0)}$, $B_1^{(i)}$ and $B_3^{(i)}$ as

$$\epsilon_i^3(\eta_i+2)(\tfrac{1}{2}B_1^{(i)}+B_4^{(i)}) = \sum_{r=1}^{i}\{\epsilon_{r-1}^3(\eta_{r-1}+2)(\tfrac{1}{6}+B_3^{(r-1)})+\epsilon_r^3(\eta_r+2)(\tfrac{1}{6}+B_3^{(r)})\}. \tag{4.49}$$

This leaves $B_2^{(0)}$ and $B_3^{(0)}$ to be determined. One relation between these constants can be deduced from (4.45). By setting $i = N$ in (4.48), and using (4.45), we obtain

$$2\mu_N\epsilon_N^2(\eta_N+1)B_1^{(N)}+\mu_0\epsilon_0^2\{(\eta_0+1)+\eta_0 B_2^{(0)}+(\eta_0+2)B_3^{(0)}\}$$
$$+2\sum_{r=1}^{N}\{\mu_{r-1}\epsilon_{r-1}^2(\eta_{r-1}+1)B_1^{(r-1)}+\mu_r\epsilon_r^2(\eta_r+1)B_1^{(r)}\} = 0. \tag{4.50}$$

For the second relation between $B_2^{(0)}$ and $B_3^{(0)}$, we may impose one further condition. A natural choice is to specify that $w = \tilde{w}$ at $z = 0$; that is, to require that the mid-surface deflection coincides with the deflection of the equivalent plate. From (4.34), this requirement is assured if

$$B_2^{(0)} = 0. \tag{4.51}$$

This would appear to be the most suitable choice if edge boundary conditions are imposed on the deflection.

Alternatively, we may specify that the resultant bending moments in the laminated plate coincide with the resultant bending moments in the equivalent plate. This choice is natural if edge moments are specified as boundary conditions. The bending moments are defined by (2.25) and, from (3.12), (3.13) and (4.35), are given by

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \tilde{M}_{xx} \\ \tilde{M}_{yy} \\ \tilde{M}_{xy} \end{bmatrix} + 2[\mu_0\epsilon_0^3 h_0^2(\tfrac{1}{15}+\tfrac{2}{3}B_3^{(0)})(\eta_0+2)$$
$$+2\sum_{i=1}^{N}\epsilon_i^3\mu_i h_i\{(\tfrac{1}{15}+\tfrac{2}{3}B_3^{(i)})h_i+(\tfrac{1}{3}B_1^{(i)}+2B_4^{(i)})H_i\}(\eta_i+2)]\begin{bmatrix} \nabla^2\tilde{w}_{,xx} \\ \nabla^2\tilde{w}_{,yy} \\ \nabla^2\tilde{w}_{,xy} \end{bmatrix}, \tag{4.52}$$

where $\tilde{M}_{xx}$, $\tilde{M}_{yy}$ and $\tilde{M}_{xy}$ are bending moments for the equivalent plate and are given by (3.15). Thus in this case the required additional condition is

$$\mu_0 \epsilon_0^3 h_0^2 (\tfrac{1}{15}+\tfrac{2}{3}B_3^{(0)})(\eta_0+2)+2\sum_{i=1}^{N} \epsilon_i^3 \mu_i h_i\{(\tfrac{1}{15}+\tfrac{2}{3}B_3^{(i)})\,h_i+(\tfrac{1}{3}B_1^{(i)}+2B_4^{(i)})\,H_i\}(\eta_i+2)=0. \quad (4.53)$$

When the constants $B_\alpha^{(i)}$ have been determined, (4.33)–(4.36) give exact three-dimensional solutions for bending deformations of the laminated plate, provided only that $\tilde{w}$ satisfies the biharmonic equation in two dimensions.

From (4.36), the difference between $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_{i+1}$, where $\boldsymbol{\tau}_i$ denotes the interlaminar shear traction at the interface between layer $i-1$ and layer i, is

$$\begin{aligned}\boldsymbol{\tau}_{i+1}-\boldsymbol{\tau}_i &= 4\epsilon_i^2\mu_i B_1^{(i)}(\eta_i+1)\,\mathrm{grad}\,\nabla^2\tilde{w},\\ &= 4a^{-2}h_i H_i \mu_i(\eta_i+1)\,\mathrm{grad}\,\nabla^2\tilde{w}.\end{aligned}$$

Hence, because the faces of the laminate are free from traction, we have

$$a^2\boldsymbol{\tau}_i = -4\sum_{r=i}^{N} h_r H_r \mu_r(\eta_r+1)\,\mathrm{grad}\,\nabla^2\tilde{w}. \quad (4.54)$$

It follows that $\boldsymbol{\tau}_i$ has its greatest magnitude at $i=1$, that is, at the interfaces closest to the mid-plane of the plate.

It now follows from (4.36), (4.46) and (4.54) that the shear stress components $\sigma_{xz}^{(i)}$ and $\sigma_{yz}^{(i)}$ can be expressed explicitly as

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma_{xz}^{(i)}\\ \sigma_{yz}^{(i)}\end{bmatrix} = \left\{\epsilon_i^2(z_i+1)\left[z_i-1+\frac{2H_i}{h_i}\right](\eta_i+1)-\frac{4}{\mu_i}\sum_{r=i}^{N}\epsilon_r^2\frac{H_r}{h_r}\mu_r(\eta_r+1)\right\}\begin{bmatrix}\nabla^2\tilde{w}_{,x}\\ \nabla^2\tilde{w}_{,y}\end{bmatrix}. \quad (4.55)$$

5. Examples

(a) *Stretching of a laminated plate containing a circular hole*

As an illustration of the theory described in §4*a*, we consider a laminated plate containing a traction-free circular hole and subjected to uniaxial tension at infinity. The solution of this problem for a homogeneous plate is well known (see, for example, Timoshenko & Goodier 1951). This solution is chosen as the equivalent plate solution. In terms of plane polar coordinates (r, θ) with origin at the centre of the hole, the solution is

$$\left.\begin{aligned}\hat{u}_r &= \frac{P}{4\hat{\mu}}\left\{\frac{R}{2\hat{\eta}+1}+\frac{1}{R}+\left[R+\frac{4(\hat{\eta}+1)}{2\hat{\eta}+1}\frac{1}{R}-\frac{1}{R^3}\right]\cos 2\theta\right\},\\ \hat{u}_\theta &= -\frac{P}{4\hat{\mu}}\left\{R+\frac{2}{2\hat{\eta}+1}\frac{1}{R}+\frac{1}{R^3}\right\}\sin 2\theta,\end{aligned}\right\} \quad (5.1)$$

where $a\hat{u}_r$ and $a\hat{u}_\theta$ denote the components of displacement in the r and θ directions respectively,

a is the radius of the hole, $R = r/a$, and P is the magnitude of the uniaxial tension applied in the $\theta = 0$ direction as $r \to \infty$. The corresponding stress components are

$$\left.\begin{aligned} \hat{\sigma}_{rr} &= \tfrac{1}{2}P\left\{1-\frac{1}{R^2}+\left[1-\frac{4}{R^2}+\frac{3}{R^4}\right]\cos 2\theta\right\}, \\ \hat{\sigma}_{\theta\theta} &= \tfrac{1}{2}P\left\{1+\frac{1}{R^2}-\left[1+\frac{3}{R^4}\right]\cos 2\theta\right\}, \\ \hat{\sigma}_{r\theta} &= -\tfrac{1}{2}P\left[1+\frac{2}{R^2}-\frac{3}{R^4}\right]\sin 2\theta. \end{aligned}\right\} \tag{5.2}$$

In terms of $\hat{u}_r$ and $\hat{u}_\theta$, we have

$$\hat{\Delta} = \frac{\partial \hat{u}_r}{\partial R}+\frac{\hat{u}_r}{R}+\frac{1}{R}\frac{\partial \hat{u}_\theta}{\partial \theta} \tag{5.3}$$

which, for the displacement (5.1), gives

$$\hat{\Delta} = \frac{P}{2\hat{\mu}(2\hat{\eta}+1)}\left[1-\frac{2}{R^2}\cos 2\theta\right]. \tag{5.4}$$

The stress and displacement in the laminate are then given by substituting (5.4) into (4.2) and (4.6)–(4.8), and then transforming into components referred to plane polar coordinates, as

$$\begin{bmatrix} u_r^{(i)} \\ u_\theta^{(i)} \end{bmatrix} = \begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \end{bmatrix} - \frac{2\epsilon_i^2 P}{R^3\hat{\mu}(2\hat{\eta}+1)}\left(\tfrac{1}{2}z_i^2+S_2^{(i)}z_i+S_3^{(i)}\right)(\eta_i-2\hat{\eta})\begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}, \tag{5.5}$$

$$w^{(i)} = -\frac{\epsilon_i\eta_i(z_i+S_1^{(i)})P}{2\hat{\mu}(2\hat{\eta}+1)}\left[1-\frac{2}{R^2}\cos 2\theta\right],$$

$$\frac{1}{\mu_i}\begin{bmatrix} \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \sigma_{r\theta}^{(i)} \end{bmatrix} = \frac{1}{\hat{\mu}}\begin{bmatrix} \hat{\sigma}_{rr} \\ \hat{\sigma}_{\theta\theta} \\ \hat{\sigma}_{r\theta} \end{bmatrix} + \frac{\eta_i-\hat{\eta}}{\hat{\mu}(2\hat{\eta}+1)}\begin{bmatrix} \hat{\sigma}_{rr}+\hat{\sigma}_{\theta\theta} \\ \hat{\sigma}_{rr}+\hat{\sigma}_{\theta\theta} \\ 0 \end{bmatrix}$$
$$+\frac{12\epsilon_i^2(\eta_i-2\hat{\eta})P}{R^4\hat{\mu}(2\hat{\eta}+1)}\left(\tfrac{1}{2}z_i^2+S_2^{(i)}z_i+S_3^{(i)}\right)\begin{bmatrix} \cos 2\theta \\ -\cos 2\theta \\ \sin 2\theta \end{bmatrix}, \tag{5.6}$$

$$\frac{1}{\mu_i}\begin{bmatrix} \sigma_{rz}^{(i)} \\ \sigma_{\theta z}^{(i)} \end{bmatrix} = -\frac{2\epsilon_i P}{R^3\hat{\mu}(2\hat{\eta}+1)}\left\{2(\eta_i-\hat{\eta})z_i+\eta_i S_1^{(i)}+S_2^{(i)}(\eta_i-2\hat{\eta})\right\}\begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}. \tag{5.7}$$

To determine the constants $\hat{\mu}$ and $\hat{\eta}$ of the equivalent plate it is necessary to specify the geometrical and mechanical properties of the laminate. For illustration we consider the simplest case of a laminate comprised of three layers each of equal thickness $2h_0$, the inner layer having elastic constants λ_0', μ_0, and the two outer layers having elastic constants λ_1', μ_1. The analysis is readily extended to plates with an arbitrary number of laminae, but for five or more laminae the algebraic expressions for the constants are complicated and are not meaningful, and it becomes better to proceed by numerical evaluation of the constants. This can be done in a straightforward way by using the recurrence relations of §4.

For the three-layer laminate, we have from (3.1)

$$\eta_0 = \lambda_0'/2\mu_0, \quad \eta_1 = \lambda_1'/2\mu_1, \tag{5.8}$$

from (3.5) and (3.6) with $N = 1$

$$\epsilon_0 = \epsilon_1 = h_0/a, \quad \epsilon = 3h_0/a, \tag{5.9}$$

and from (3.7) and (3.8)

$$\hat{\lambda}' = \tfrac{1}{3}(\lambda_0' + 2\lambda_1'), \quad \hat{\mu} = \tfrac{1}{3}(\mu_0 + 2\mu_1), \quad \hat{\eta} = \frac{\hat{\lambda}'}{2\hat{\mu}} = \frac{\lambda_0' + 2\lambda_1'}{2(\mu_0 + 2\mu_1)}. \tag{5.10}$$

It remains to determine the constants $S_1^{(0)}$, $S_2^{(0)}$, $S_3^{(0)}$, $S_1^{(1)}$, $S_2^{(1)}$ and $S_3^{(1)}$. From (4.12)

$$S_1^{(0)} = 0, \quad S_2^{(0)} = 0, \tag{5.11}$$

and from (4.21)–(4.23), with $r = 1$,

$$\eta_1 S_1^{(1)} = \eta_0 + \eta_1, \tag{5.12}$$

$$\mu_1(\eta_1 - 2\hat{\eta})\, S_2^{(1)} = 2\mu_0(\eta_0 - \hat{\eta}) - \mu_1(\eta_0 - \eta_1 + 2\hat{\eta}), \tag{5.13}$$

$$(\eta_1 - 2\hat{\eta})\, S_3^{(1)} = (\eta_0 - 2\hat{\eta})\, S_3^{(0)} + \tfrac{1}{2}(\eta_0 - \eta_1) + (\eta_1 - 2\hat{\eta})\, S_2^{(1)}. \tag{5.14}$$

Because stress boundary conditions are prescribed at $r = a$, we adopt (4.29) or (4.30) to determine $S_3^{(0)}$. From (4.29)

$$\mu_0(S_3^{(0)} + \tfrac{1}{6})(\eta_0 - 2\hat{\eta}) + 2\mu_1(S_3^{(1)} + \tfrac{1}{6})(\eta_1 - 2\hat{\eta}) = 0. \tag{5.15}$$

Hence, from (5.14) and (5.15),

$$(\mu_0 + 2\mu_1)(\eta_0 - 2\hat{\eta})\, S_3^{(0)} = -2\mu_1(\eta_1 - 2\hat{\eta})\, S_2^{(1)} - \mu_1(\eta_0 - \eta_1) - \tfrac{1}{6}\{\mu_0(\eta_0 - 2\hat{\eta}) + 2\mu_1(\eta_1 - 2\hat{\eta})\}, \tag{5.16}$$

$$(\mu_0 + 2\mu_1)(\eta_1 - 2\hat{\eta})\, S_3^{(1)} = \mu_0(\eta_1 - 2\hat{\eta})\, S_2^{(1)} + \tfrac{1}{2}\mu_0(\eta_0 - \eta_1) - \tfrac{1}{6}\{\mu_0(\eta_0 - 2\hat{\eta}) + 2\mu_1(\eta_1 - 2\hat{\eta})\}.$$

In the equivalent plate solution, the deviation of the stress from uniform tension decays as R^{-2}. From (5.5)–(5.7) we see that in the exact solution σ_{rz} and $\sigma_{\theta z}$ decay as R^{-3}, while the correction to the equivalent plate displacement also decays as R^{-3}, and the corrections to the in-plane stress components decay as R^{-4}. From (5.7) with $i = 0$, $z_0 = 1$, it follows that the magnitude of the shear traction at the interfaces is the magnitude of

$$\tau_1 = \frac{4\epsilon_0 \mu_0(\eta_0 - \hat{\eta})\, P}{R^3 \hat{\mu}(2\hat{\eta} + 1)}. \tag{5.17}$$

As Poisson's ratio ν_i varies from 0 to $\frac{1}{2}$, so η_i varies from 0 to 1. The extreme values of τ_1 occur when (a) $\eta_0 = 0$, $\eta_1 = 1$, $\hat{\eta} = \frac{2}{3}\mu_1/\hat{\mu}$, and (b) $\eta_0 = 1$, $\eta_1 = 0$, $\hat{\eta} = \frac{1}{3}\mu_0/\hat{\mu}$. The corresponding values of τ_1 are

$$(a) \quad -\frac{8\epsilon_0 \mu_0 \mu_1 P}{R^3 \hat{\mu}(\mu_0 + 6\mu_1)}, \qquad (b) \quad \frac{8\epsilon_0 \mu_0 \mu_1 P}{R^3 \hat{\mu}(3\mu_0 + 2\mu_1)}. \tag{5.18}$$

We observe that τ_1 is independent of θ.

The terms proportional to ϵ_i^2 in (5.5) give the additional in-plane displacement of the exact theory compared with the classical laminate theory. The magnitude of this additional displacement is independent of θ. Similarly the terms proportional to ϵ_i^2 in (5.6) represent the corrections to the in-plane stress components due to replacing classical laminate theory by the exact theory. We note, however, that no correction is needed for $\sigma_{rr}+\sigma_{\theta\theta}$.

(b) *Bending of a laminated plane containing a circular hole*

As an illustration of the bending solutions described in §4b, we again consider a plate containing a circular hole of radius a, but now suppose it to be bent by couples remote from the hole such that

$$M_{xx}\to M, \quad M_{yy}\to 0 \quad \text{as} \quad R\to\infty, \tag{5.19}$$

where we use the same plane polar coordinate system (r,θ) and scaled coordinate $R=r/a$ as in §5a. The boundary of the hole is supposed to be subject to zero resultant bending couple and vertical force, so that (Timoshenko & Woinowsky-Krieger 1959)

$$M_{rr}=0, \quad N_{rz}-(1/a)\,\partial M_{r\theta}/\partial\theta=0, \quad R=1. \tag{5.20}$$

The equivalent plate solution is taken to be the solution of classical thin plate theory which satisfies (2.31) and the boundary conditions (5.19) and (5.20). This solution (Timoshenko & Woinowsky-Krieger 1959) is

$$\tilde{w}=-\frac{3M}{16\tilde{\mu}\epsilon H^2}\left\{2\ln R+\frac{R^2}{2\tilde{\eta}+1}+\cos 2\theta\left[R^2+\frac{2}{4\tilde{\eta}+3}-\frac{1}{4\tilde{\eta}+3}\frac{1}{R^2}\right]\right\}. \tag{5.21}$$

The corresponding stress in the equivalent plate (expressed in components referred to (r,θ,z) coordinates) is

$$\left.\begin{aligned}
\tilde{\sigma}_{rr}&=\frac{3Mz}{4H^2}\left\{1-\frac{1}{R^2}+\cos 2\theta\left[1-\frac{4\tilde{\eta}}{4\tilde{\eta}+3}\frac{1}{R^2}-\frac{3}{4\tilde{\eta}+3}\frac{1}{R^4}\right]\right\},\\
\tilde{\sigma}_{\theta\theta}&=\frac{3Mz}{4H^2}\left\{1+\frac{1}{R^2}-\cos 2\theta\left[1+\frac{4\tilde{\eta}}{4\tilde{\eta}+3}\frac{1}{R^2}-\frac{3}{4\tilde{\eta}+3}\frac{1}{R^4}\right]\right\},\\
\tilde{\sigma}_{r\theta}&=-\frac{3Mz}{4H^2}\sin 2\theta\left\{1-\frac{1}{4\tilde{\eta}+3}\left[\frac{2}{R^2}-\frac{3}{R^4}\right]\right\},
\end{aligned}\right\} \tag{5.22}$$

$$\begin{bmatrix}\tilde{\sigma}_{rz}\\ \tilde{\sigma}_{\theta z}\end{bmatrix}=\frac{3\epsilon M(\tilde{\eta}+1)}{H^2(4\tilde{\eta}+3)}(1-z^2)\begin{bmatrix}\cos 2\theta\\ \sin 2\theta\end{bmatrix}, \tag{5.23}$$

and the in-plane displacement components for the equivalent plate are

$$\begin{bmatrix}\tilde{u}_r\\ \tilde{u}_\theta\end{bmatrix}=\frac{3Mz}{8\tilde{\mu}H^2}\begin{bmatrix}\dfrac{R}{2\tilde{\eta}+1}+\dfrac{1}{R}+\cos 2\theta\left[R+\dfrac{1}{4\tilde{\eta}+3}\dfrac{1}{R^3}\right]\\ -\sin 2\theta\left\{R+\dfrac{1}{4\tilde{\eta}+3}\left[\dfrac{2}{R}-\dfrac{1}{R^3}\right]\right\}\end{bmatrix}. \tag{5.24}$$

From (5.21), we have

$$\nabla^2\tilde{w}=-\frac{3M}{4\tilde{\mu}\epsilon H^2}\left[\frac{1}{2\tilde{\eta}+1}-\frac{2\cos 2\theta}{4\tilde{\eta}+3}\frac{1}{R^2}\right]. \tag{5.25}$$

The displacement and stress components in each layer are now given by substituting (5.21) and (5.25) into (4.33)–(4.36), and transforming to cylindrical polar coordinates, as

$$\begin{bmatrix}u_r^{(i)}\\ u_\theta^{(i)}\end{bmatrix}=\begin{bmatrix}\tilde{u}_r\\ \tilde{u}_\theta\end{bmatrix}-\frac{3M\epsilon_i^3(\eta_i+2)}{\tilde{\mu}\epsilon H^2(4\tilde{\eta}+3)R^3}\left(\tfrac{1}{6}z_i^3+\tfrac{1}{2}B_1^{(i)}z_i^2+B_3^{(i)}z_i+B_4^{(i)}\right)\begin{bmatrix}\cos 2\theta\\ \sin 2\theta\end{bmatrix}, \tag{5.26}$$

$$w^{(i)} = \tilde{w} - \frac{3M\epsilon_i^2\eta_i}{4\tilde{\mu}\epsilon H^2}\left(\tfrac{1}{2}z_i^2 + B_1^{(i)}z_i + B_2^{(i)}\right)\left[\frac{1}{2\tilde{\eta}+1} - \frac{2\cos 2\theta}{4\tilde{\eta}+3}\frac{1}{R^2}\right], \tag{5.27}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma_{rr}^{(i)}\\ \sigma_{\theta\theta}^{(i)}\\ \sigma_{r\theta}^{(i)}\end{bmatrix} = \frac{1}{\tilde{\mu}}\begin{bmatrix}\tilde{\sigma}_{rr}\\ \tilde{\sigma}_{\theta\theta}\\ \tilde{\sigma}_{r\theta}\end{bmatrix} + \frac{1}{\tilde{\mu}}\left[\frac{\eta_i - \tilde{\eta}}{2\tilde{\eta}+1}\right]\begin{bmatrix}\tilde{\sigma}_{rr}+\tilde{\sigma}_{\theta\theta}\\ \tilde{\sigma}_{rr}+\tilde{\sigma}_{\theta\theta}\\ 0\end{bmatrix}$$

$$+\frac{18M\epsilon_i^3(\eta_i+2)}{\tilde{\mu}\epsilon H^2(4\tilde{\eta}+3)R^4}\left(\tfrac{1}{6}z_i^3 + \tfrac{1}{2}B_1^{(i)}z_i^2 + B_3^{(i)}z_i + B_4^{(i)}\right)\begin{bmatrix}\cos 2\theta\\ -\cos 2\theta\\ \sin 2\theta\end{bmatrix}, \tag{5.28}$$

$$\frac{1}{\mu_i}\begin{bmatrix}\sigma_{rz}^{(i)}\\ \sigma_{\theta z}^{(i)}\end{bmatrix} = -\frac{3M\epsilon_i^2}{\tilde{\mu}\epsilon H^2(4\tilde{\eta}+3)R^3}\{2(\tfrac{1}{2}z_i^2 + B_1^{(i)}z_i)(\eta_i+1) + \eta_i B_2^{(i)} + (\eta_i+2)B_3^{(i)}\}\begin{bmatrix}\cos 2\theta\\ \sin 2\theta\end{bmatrix}. \tag{5.29}$$

Alternatively, $\sigma_{rz}^{(i)}$ and $\sigma_{\theta z}^{(i)}$ can be expressed independently of the constants $B_\alpha^{(i)}$ by means of (4.55).

To specify the elastic constants of the equivalent plate we adopt the same laminate geometry as in §5*a*, namely, a symmetric laminate of three laminae of equal thickness having elastic constants λ_0', μ_0 and λ_1', μ_1 in the inner and outer layers respectively. Then, from (3.12),

$$\tilde{\lambda}' = \tfrac{1}{27}(\lambda_0' + 26\lambda_1'), \quad \tilde{\mu} = \tfrac{1}{27}(\mu_0 + 26\mu_1), \quad \tilde{\eta} = \tilde{\lambda}'/2\tilde{\mu}. \tag{5.30}$$

The constants $B_\alpha^{(0)}$, $B_\alpha^{(1)}$ ($\alpha = 1, 2, 3, 4$) are determined as follows. From (4.40),

$$B_1^{(0)} = 0, \quad B_4^{(0)} = 0. \tag{5.31}$$

From (4.41)–(4.45),

$$\left.\begin{aligned} B_1^{(1)} = 2, \quad \eta_0(\tfrac{1}{2}+B_2^{(0)}) &= \eta_1(-\tfrac{3}{2}+B_2^{(1)}),\\ \mu_0\{(\eta_0+1)+\eta_0 B_2^{(0)} + (\eta_0+2)B_3^{(0)}\} &= \mu_1\{-3(\eta_1+1)+\eta_1 B_2^{(1)} + (\eta_1+2)B_3^{(1)}\},\\ 5(\eta_1+1)+\eta_1 B_2^{(1)} + (\eta_1+2)B_3^{(1)} &= 0,\\ (\eta_0+2)(\tfrac{1}{6}+B_3^{(0)}+B_4^{(0)}) &= (\eta_1+2)(\tfrac{5}{6}-B_3^{(1)}+B_4^{(1)}). \end{aligned}\right\} \tag{5.32}$$

As the final condition, we impose the requirement (4.53) that the resultant bending moments in the laminate coincide with those in the equivalent plate. This gives

$$\mu_0(\eta_0+2)(\tfrac{1}{15}+\tfrac{2}{3}B_3^{(0)}) + 2\mu_1(\eta_1+2)(\tfrac{7}{5}+\tfrac{2}{3}B_3^{(1)}+4B_4^{(1)}) = 0. \tag{5.33}$$

These equations determine all the constants in terms of the material properties, and so complete the solution.

The shear stress components σ_{rz} and $\sigma_{\theta z}$ are most conveniently obtained directly from (4.55) and (5.25) as

$$\left.\begin{aligned} \begin{bmatrix}\sigma_{rz}^{(0)}\\ \sigma_{\theta z}^{(0)}\end{bmatrix} &= -\frac{M\epsilon\{\mu_0(\eta_0+1)(z_0^2-1) - 8\mu_1(\eta_1+1)\}}{3\tilde{\mu}H^2(4\tilde{\eta}+3)R^3}\begin{bmatrix}\cos 2\theta\\ \sin 2\theta\end{bmatrix},\\ \begin{bmatrix}\sigma_{rz}^{(1)}\\ \sigma_{\theta z}^{(1)}\end{bmatrix} &= -\frac{M\mu_1\epsilon(\eta_1+1)(z_1-1)(z_1+5)}{3\tilde{\mu}H^2(4\tilde{\eta}+3)R^3}\begin{bmatrix}\cos 2\theta\\ \sin 2\theta\end{bmatrix}. \end{aligned}\right\} \tag{5.34}$$

The solution described in this section has many features that are similar to those of the analogous stretching problem which was described in §5*a*. The leading terms in the expressions for u_r, u_θ, w, σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ coincide with the expressions given by classical laminate theory. The additional terms may be regarded as corrections to that theory. We note that the corrections to u_r and u_θ decay as R^{-3}, and those to σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ as R^{-4}. The magnitude of the correction to the in-plane displacement is independent of θ, and the correction to $\sigma_{rr}+\sigma_{\theta\theta}$ is identically zero everywhere.

Numerical investigation shows that, in the main, the results of this section are not sensitive to the values of η_0 and η_1. For simplicity, we consider the case $\eta_0 = \eta_1 = \tilde{\eta}$. Then the difference between the values of the in-plane stress components given by the present theory and the corresponding values given by classical laminate theory is, from (5.28),

$$\begin{bmatrix}\sigma^{(i)}_{rr}\\ \sigma^{(i)}_{\theta\theta}\\ \sigma^{(i)}_{r\theta}\end{bmatrix} - \frac{\mu_i}{\tilde{\mu}}\begin{bmatrix}\tilde{\sigma}_{rr}\\ \tilde{\sigma}_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix} = \sigma_i\begin{bmatrix}\cos 2\theta\\ -\cos 2\theta\\ \sin 2\theta\end{bmatrix},$$

where
$$\sigma_i = \frac{18M\mu_i\epsilon_i^3(\eta_i+2)\,(\frac{1}{6}z_i^3+\frac{1}{2}B_1^{(i)}z_i^2+B_3^{(i)}z_i+B_4^{(i)})}{\tilde{\mu}\epsilon H^2(4\tilde{\eta}+3)\,R^4}.$$

Table 1 gives values of $\sigma_i H^2/M\epsilon^2$ for $\eta_i = \frac{3}{7}$ (which corresponds to Poisson's ratio $\nu_i = 0.3$) and $\mu_0/\mu_1 = 0.1$, 1.0 and 10 at $R = 1$, at several values of z. A test of the accuracy of classical laminate theory is to compare the values of σ_i with the stress given by classical laminate theory

Table 1. Values of $\sigma_i H^2/M\epsilon^2$, showing variation of the higher-order in-plane stress amplitudes through a three-layer symmetric laminate with $\epsilon_0 = \epsilon_1 = \frac{1}{3}\epsilon$ and $\nu_0 = \nu_1 = 0.3$, evaluated at $R = 1$ for various values of μ_0/μ_1

μ_0/μ_1	0.1	1.0	10.0
mid-plane, $z = 0$, $z_0 = 0$	0	0	0
interface, $z = \frac{1}{3}$, $z_0 = 1$	−0.83	−0.25	2.87
interface, $z = \frac{1}{3}$, $z_1 = -1$	−8.34	−0.25	0.29
mid-lamina, $z = \frac{2}{3}$, $z_1 = 0$	−1.28	−0.16	−0.26
free-surface, $z = 1$, $z_1 = 1$	6.49	0.62	−0.29

which, in the ith layer, is of order $\mu_i M/H^2\tilde{\mu}$. From table 1 we find that when $\mu_0/\mu_1 = 0.1$, then $H^2\tilde{\mu}\sigma_i/\mu_i M$ varies between $-8.0\epsilon^2$ and $6.3\epsilon^2$, but in the cases $\mu_0/\mu_1 = 1.0$ and $\mu_0/\mu_1 = 10.0$, then $H^2\tilde{\mu}\sigma_i/\mu_i M$ is everywhere less than ϵ^2 in magnitude. Consequently, in this respect, the classical laminate theory appears to give a valid approximation over a wide range of values of the elastic constants, provided that $\epsilon^2 \ll 1$.

Classical laminate theory gives no information about the shear stress components σ_{rz} and $\sigma_{\theta z}$. In the exact theory these are given by (5.34). We observe that these components decay as R^{-3}, and that the magnitude $\tau = (\sigma_{rz}^2+\sigma_{\theta z}^2)^{\frac{1}{2}}$ of the shear traction is independent of θ. In figure 2 we show the variation with z of $\tau H^2/M\epsilon$ at $R = 1$, with $\eta_0 = \eta_1 = \frac{3}{7}$, and $\mu_0/\mu_1 = 0.1$, 1.0, 10.

As an example of a problem in which displacement boundary conditions are imposed, we consider briefly a plate containing a circular hole subject to the same conditions (5.19) as $R \to \infty$, but clamped at $R = 1$, so that

$$\tilde{w} = 0, \quad \partial\tilde{w}/\partial r = 0 \quad \text{at} \quad R = 1. \tag{5.35}$$

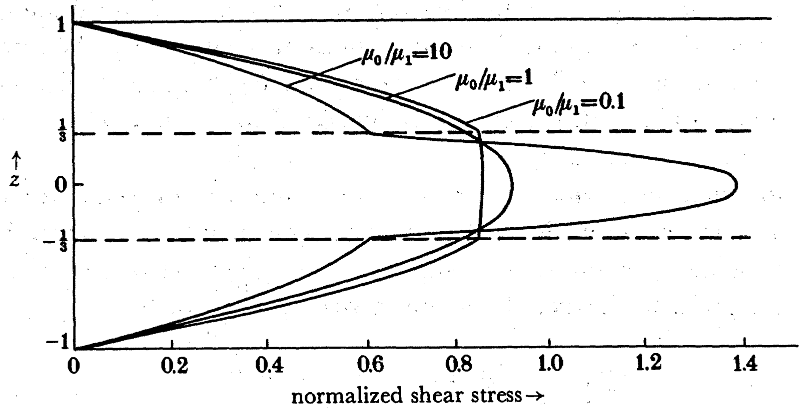

FIGURE 2. Variation of the amplitude of $H^2\sigma_{rz}/M\epsilon$ and $H^2\sigma_{\theta z}/M\epsilon$ through a three-layer symmetric laminate with $\epsilon_0 = \epsilon_1 = \frac{1}{3}\epsilon$ and $\nu_0 = \nu_1 = 0.3$, evaluated at $R = 1$.

The corresponding equivalent plate solution is

$$\tilde{w} = -\frac{3M}{16\tilde{\mu}\epsilon H^2}\left\{\frac{R^2-1-2\ln R}{2\tilde{\eta}+1}+\left[R^2-2+\frac{1}{R^2}\right]\cos 2\theta\right\}. \tag{5.36}$$

Hence, in this case,

$$\nabla^2\tilde{w} = -\frac{3M}{4\tilde{\mu}\epsilon H^2}\left[\frac{1}{2\tilde{\eta}+1}+\frac{2}{R^2}\cos 2\theta\right]. \tag{5.37}$$

Consequently, $\tilde{w}$ and $\nabla^2\tilde{w}$ have the same forms as in (5.21) and (5.25), but with different values of the coefficients. The expressions for the displacement and stress components are therefore of the same forms as in the previous problem, with appropriately modified coefficients. For the displacement boundary value problem it is preferable to adopt (4.51) rather than (4.53) to determine $B_2^{(0)}$; therefore the condition (5.33) is replaced by

$$B_2^{(0)} = 0. \tag{5.38}$$

The remaining analysis and discussion follow straightforwardly as in the previous problem, and we omit the details.

6. Discussion of the exact laminate theory

As was stated previously, when the constants $S_\alpha^{(i)}$, $B_\beta^{(i)}$ ($\alpha = 1, 2, 3;\ \beta = 1, 2, 3, 4;\ i = 0, 1, \ldots, N$) have been determined in the manner described in §4, the solutions given in that section are exact three-dimensional solutions of the equations of isotropic linear elasticity. In order to construct such solutions it is only necessary (*a*) to determine an appropriate solution of the two-dimensional thin-plate equations (the solution for the 'equivalent plate') and (*b*) evaluate the constants $S_\alpha^{(i)}$ and $B_\beta^{(i)}$. These constants depend only on the thicknesses of the laminae and on the elastic constants of the materials which form the laminae, and are completely independent of any particular solution under consideration. Thus, for any particular laminate with given lay-up and elastic properties, the constants may be evaluated once and for all. Explicit formulae for the $S_\alpha^{(i)}$ and $B_\beta^{(i)}$ are given in §4. Although these formulae appear cumbersome when they are expressed algebraically, especially for laminates with a large number of laminae, the numerical evaluation of the constants by using the recurrence relations which they satisfy is very simple and straightforward.

Having determined the equivalent plate solution and the constants $S_\alpha^{(i)}$ and $B_\beta^{(i)}$, it becomes a matter of mere substitution in the formulae of §4 to write down the corresponding exact three-dimensional solution. The solutions given in §5 show the simplicity of the procedure. Even if the equivalent plate solution is a numerical one (obtained, for example, by finite element or finite difference methods) the three-dimensional solution can be constructed from it by appropriate numerical differentiations of the displacement given by the equivalent plate solution.

In the exact solution, the leading terms in ϵ in the displacement and the in-plane stress components are identical to the corresponding quantities given by classical laminate theory. The remaining terms may be regarded as a correction to classical laminate theory, and are a measure of the accuracy of that theory. In addition, the exact theory yields the values of the shear stress components σ_{xz} and σ_{yz}, about which classical laminate theory gives no information, and in particular of the inter-laminar shear stress, which in practice may be an important factor in causing the onset of delamination.

Neither classical plate theory nor the exact theory is capable of satisfying edge boundary conditions point by point, and can only do so in an average manner. In this average sense the standard boundary value problems outlined at the end of §2c can be solved by the present theory. The equivalent plate solution is constructed, by any of the methods available for solving two-dimensional elasticity problems, to satisfy the specified boundary conditions and then, with the appropriate choice of $S_3^{(0)}$ and $B_2^{(0)}$, the three-dimensional solution will conform to the same averaged boundary conditions as the equivalent plate solution. If edge boundary conditions are specified point by point, there will be a boundary layer region adjacent to the edge in which the solution does not conform exactly to the boundary conditions; by invoking Saint-Venant's principle, it can be argued that this layer penetrates to a distance only of the order of the plate thickness.

We also observe, without presenting details, that the results of §4 can be expressed very conveniently and concisely in terms of complex variable notation. For stretching deformations the equivalent plate displacement components $\hat{u}$ and $\hat{v}$ may be expressed in the conventional way (as described, for example, by Muskhelishvili (1963), Green & Zerna (1954) or England (1971)) as

$$2\hat{\mu}(\hat{u}+\mathrm{i}\hat{v}) = \frac{3+2\hat{\eta}}{1+2\hat{\eta}}\Omega(\zeta)-\zeta\overline{\Omega'(\zeta)}-\overline{\omega(\zeta)},$$

where the complex potentials $\Omega(\zeta)$ and $\omega(\zeta)$ are analytic functions of $\zeta = x+\mathrm{i}y$, and bars here denote complex conjugates. Then $\hat{\Delta}$ and the derivatives of $\hat{u}$, $\hat{v}$ and $\hat{\Delta}$ which occur in (4.6)–(4.8) are easily expressed in terms of $\Omega(\zeta)$, $\omega(\zeta)$ and their derivatives. Similarly for bending deformations, the biharmonic function $\tilde{w}$ can be expressed as

$$2\tilde{w} = \bar{\zeta}\psi(\zeta)+\zeta\overline{\psi(\zeta)}+\phi(\zeta)+\overline{\phi(\zeta)},$$

where $\psi(\zeta)$ and $\phi(\zeta)$ are also complex potentials. The derivatives of $\tilde{w}$ which occur in (4.33)–(4.36) are readily expressed in terms of derivatives of $\psi(\zeta)$ and $\phi(\zeta)$.

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